MP- 204

Vardhaman Mahaveer Open University, Kota

Quantitative Techniques

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CONTENTS

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Unit-1 : Quantitative Techniques in Business

Unit Structure:

1.0 Objectives

After reading this unit, you should be able to understand the:

- Meaning of Quantitative Techniques
- Classification of Quantitative Techniques
- Role of Quantitative Techniques
- Limitations of Quantitative Techniques
- Functions and their applications
- Construction and Classification of functions
- Functions related to economics and some special functions
- Roots of a function
- Break Even Analysis
- Sequence and Series
- Arithmetic Progression, Sum of A.P. Series, Arithmetic Mean
- Geometric Progression, Sum of G.P. Series, Geometric Mean
- Applications of A.P. and G.P.

1.1 Introduction

Quantitative technique is a very powerful tool, by using this we can augment our production, maximize profits, minimize costs, and production methods can be oriented for the accomplishment of certain pre – determined objectives. Quantitative techniques used to solve many of the problems that arise in a business or industrial area. A large number of business problems, in the relatively recent past, have been given a quantitative representation with considerable degree of success. All this has attracted the students, business executives, public administrators alike towards the study of these techniques more and more in the present times.

Scientific methods have been man's outstanding asset to pursue an ample number of activities. It is analyzed that whenever some national crisis, emerges due to the impact of political, social, economic or cultural factors the talents from all walks of life amalgamate together to overcome the situation and rectify the problem. In this chapter we will see how the quantitative techniques had facilitated the organization in solving complex problems on time with greater accuracy. The historical development will facilitate in managerial decision-making & resource allocation, The methodology helps us in studying the scientific methods with respect to phenomenon connected with human behaviour like formulating the problem, defining decision variable and constraints, developing a suitable model, acquiring the input data, solving the model, validating the model, implementing the results. The major advantage of mathematical model is that its facilitates in taking decision faster and more accurately.

Managerial activities have become complex and it is necessary to make right decisions to avoid heavy losses. Whether it is a manufacturing unit, or a service organization, the resources have to be utilized to its maximum in an efficient manner. The future is clouded with uncertainty and fast changing, and decision-making – a crucial activity – cannot be made on a trial-and-error basis or by using a thumb rule approach. In such situations, there is a greater need for applying scientific methods to decision-making to increase the probability of coming up with good decisions. Quantitative Technique is a scientific approach to managerial decision-making. The successful use of Quantitative Technique for management would help the organization in solving complex problems on time, with greater accuracy and in the most economical way. Today, several scientific management techniques are available to solve managerial problems and use of these techniques helps managers become explicit about their objectives and provides additional information to select an optimal decision. This study material is presented with variety of these techniques with real life problem areas.

1.2 Meaning

Quantitative techniques are those statistical and operations research programming techniques which help in the decision making process especially concerning business and industry. These techniques involve the introduction of the element of quantities i.e., they involve the use of numbers, symbols and other mathematical expressions. The quantitative techniques are essentially helpful supplement to judgement and intuition. These techniques evaluate planning factors and alternative as and when they arise rather than prescribe courses of action. As such, quantitative techniques may be defined as those techniques which provide the decision maker with a systematic and powerful means of analysis and help, based on quantitative data, in exploring policies for achieving pre – determined goals. These techniques are particularly relevant to problems of complex business enterprises.

1.3 Classification of Quantitative Techniques

We have many quantitative techniques in modern times. They can broadly be put under two categories:

(a) Statistical Techniques (b) Programming Techniques

Statistical Techniques:

These techniques are used in conducting the statistical inquiry concerning a certain phenomenon. They are including all the statistical methods beginning from the collection of data till the task of interpretation of the collected data. More clearly, the methods of collection of statistical data, the technique of classification and tabulation of the collected data, the calculation of various statistical measures such as mean, standard deviation, coefficient of correlation etc, the techniques of analysis and interpretation and finally the task of deriving inference and judging their reliability are some of the important statistical techniques.

Statistical Techniques also help in Correlation too. Correlation is a statistical technique that can show whether and how strongly pairs of variables are related. For example, height and weight are related; taller people tend to be heavier than shorter people. The relationship isn't perfect. People of the same height vary in weight, and you can easily think of two people you know where the shorter one is heavier than the taller one. Nonetheless, the average weight of people 5'5'' is less than the average weight of people 5'6'', and their average weight is less than that of people 5'7'', etc. Correlation can tell you just how much of the variation in peoples' weights is related to their heights.

Although this correlation is fairly obvious your data may contain unsuspected correlations. You may also suspect there are correlations, but don't know which are the strongest. An intelligent correlation analysis can lead to a greater understanding of your data.

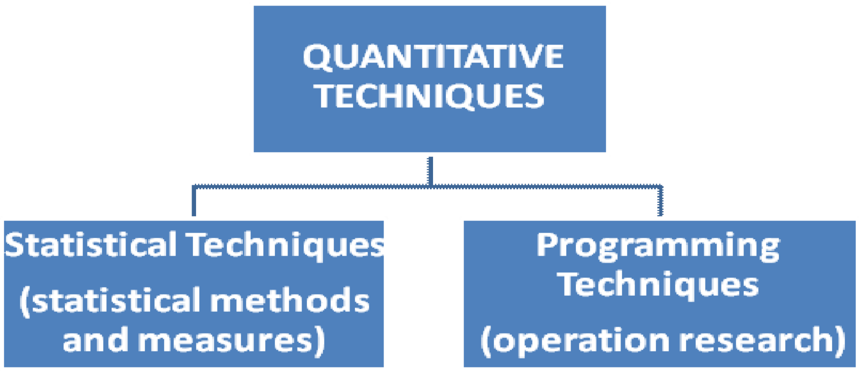
There are several different correlation techniques. The Survey System's optional Statistics Module includes the most common type, called the Pearson or product-moment correlation. The module also includes a variation on this type called partial correlation. The latter is useful when you want to look at the relationship between two variables while removing the effect of one or two other variables.

Like all statistical techniques, correlation is only appropriate for certain kinds of data. Correlation works for quantifiable data in which numbers are meaningful, usually quantities of some sort. It cannot be used for purely categorical data, such as gender, brands purchased, or favourite colour.

Programming Techniques:

It can be defined as operational research or simply (O.R.) are the model building techniques used by decision maker in modern times. They include wide variety of techniques such as linear programming, theory of games, simulation, network analysis, queuing theory and many other similar techniques.

The following chart enlists the names of the important quantitative techniques:



(i) Methods of collecting data (ii) Classification and tabulation of collected data (iii) Probability theory and Sampling Analysis (iv) Correlation and regression (v) Index number (vi) Time series Analysis (vii) Interpolation and Extrapolation (viii) Survey Techniques and Methodology (ix) Ratio Analysis (x) Statistical quality control (xi) Analysis of Variance (xii) Statistical Inferences and Interpretation (xiii) Theory of Attributes	(i) Linear Programming (ii) Decision Theory (iii) Theory of Games (iv) Simulation: a. Monte Carlo Techniques b. System Simulation (v) Waiting Line (queuing) Theory (vi) Inventory Planning (vii) Integrated Production Models (viii) Network Analysis/ PERT (ix) Others a. Non- Linear Programming b. Dynamic Programming c. Search Theory d. Integer Programming e. Quadratic Programming f. Parametric Programming g. The Theory of Replacement etc.

1.4 Role of Quantitative Techniques

These techniques are especially increasing since World War II in the technology of business administration. These techniques help in solving complex and intricate problems of business and industry. Quantitative techniques for decision making are, in fact, examples of the use of scientific method of management. Their role can be well understood under the following heads:

(i) Provide a tool for scientific analysis: These techniques provides executives with a more precise description of the cause and effect relationship and risks underlying the business operations in measurable terms and this eliminates the conventional intuitive and subjective basis on which managements used to formulate their decisions decades ago. In fact, these techniques replace the intuitive and subjective approach of decision making by an analytical and objective approach. The use of these techniques has transformed the conventional techniques of operational and investment problems in business and industry. Quantitative techniques thus encourage and enforce disciplined thinking about organisational problems.

(ii) Provide solution for various business problems: These techniques are being used in the field of production, procurement, marketing, finance and allied fields. Problems like, how best can the managers and executives allocate the available resources to various products so that in a given time the profits are maximum or the cost is minimum? Is it possible for an industrial enterprise to arrange the time and quantity of orders of its stock such that the overall profit with given resources is maximum? How far is it within the competence of a business manager to determine the number of men and machines to be employed and used in such a manner that neither remains idle and at the same time the customer or the public has not to wait unduly long for service? And similar other problems can be solved with the help of quantitative techniques.

(iii) Enable proper deployment of resources: It render valuable help in proper deployment of resources. For example, PERT enables us to determine the earliest and the latest times for each of the events and activities and thereby helps in identification of the critical path. All this helps in the deployment of the resources from one activity to another to enable the project completion on time. This techniques, thus, provides for determining the probability of completing an event or project itself by a specified date.

(iv) Helps in minimizing waiting and servicing costs: This theory helps the management men in minimizing the total waiting and servicing costs. This technique also analyses the feasibility of adding facilities and thereby helps the business people to take a correct and profitable decision.

(v) Assists in choosing an optimum strategy: Game theory is especially used to determine the optimum strategy in a competitive situation and enables the businessmen to maximise profits or minimize losses by adopting the optimum strategy.

(vi) They render great help in optimum resource allocation: Linear programming technique is used to allocate scarce resources in an optimum manner in problem of scheduling, product – mix and so on.

(vii) Enable the management to decide when to buy and how much to buy: The techniques of inventory planning enables the management to decide when to buy and how much to buy.

(viii) They facilitate the process of decision making: Decision theory enables the businessmen to select the best course of action when information is given to probabilistic form. Through decision tree techniques executive's judgement can systematically be brought into the analysis of the problems. Simulation is an other important technique used to imitate an operation or process prior to actual performance. The significance of simulation lies in the fact that it enables in finding out the effect of alternative courses of action in situation involving uncertainty where mathematical formulation is not possible. Even complex groups of variables can be handled through this technique.

(ix) Through various quantitative techniques management can know the reactions of the integrated business systems: The Integrated Production Models techniques are used to minimise cost with respect to work force, production and inventory. This technique is quite complex and is usually used by companies having detailed information concerning their sales and costs statistics over a long period. Besides, various other O.R. techniques also help in management people taking decisions concerning various problems of business and industry. The techniques are designed to investigate how the integrated business system would react to variations in its component elements and/or external factors.

1.5 Limitations

Quantitative techniques though are a great aid to management but still they cannot be substitute for decision making. The choice of criterion as to what is actually best for the business enterprise is still that of an executive who has to fall back upon his experience and judgement. This is so because of the several limitations of quantitative techniques. Important limitations of these techniques are as given below:

(i) The inherent limitation concerning mathematical expressions: Quantitative techniques involve the use of mathematical models, equations and similar other mathematical expressions. Assumptions are always incorporated in the derivation of an equation and such an equation may be correctly used for the solution of the business problems when the underlying assumptions and variables in the model are present in the concerning problem. IF this caution is not given due care then there always remains the possibility of wrong application of the quantitative techniques. Quite often the operations researchers have been accused of having many solutions without being able to find problems that fit.

(ii) High costs are involved in the use of quantitative techniques: Quantitative techniques usually prove very expensive. Services of specialised persons are invariably called for while using quantitative techniques. Even in big business organisations we can expect that quantitative techniques will continue to be of limited use simply because they are not in many cases worth their cost. As opposed to this a typical manager, exercising intuition and judgement, may be able to make a decision very inexpensively. Thus, the use of quantitative techniques is a costlier affair and this in fact constitutes a big and important limitation of such techniques.

(iii) Quantitative techniques do not take into consideration the intangible factors i.e., non measurable human factors: Quantitative techniques make no allowances for intangible factors such as skill, attitude, vigour of the management people in taking decisions but in many instances success or failure hinges upon the consideration of such non-measurable intangible factors. There cannot be any magic formula for getting an answer to management problems; much depends upon proper managerial attitudes and policies.

(iv) Quantitative techniques are just the tools of analysis and not the complete decision making process: It should always be kept in mind that quantitative techniques, whatsoever it may be, alone cannot make the final decision. They are just tools and simply suggest best alternatives but in final analysis many business decisions will involve human element. Thus, quantitative analysis is at best a supplement rather than, a substitute for management; subjective judgement is likely to remain a principal approach to decision making.

1.6 Functions and Their Applications

Functions:

A function is a technical term used to symbolise relationship between variables. When one or more independent variables are related to dependent variables then the dependent variable is said to be the function of independent variables. A function explains the nature of relationship between dependent and independent variables. A relationship may be a formula or a graph or a mathematical equation.

Examples of this type of relationship are 'price of a commodity and its demand', 'advertising expenses and sales', 'income and expenditure', etc. This relationship is denoted by the concept of a *function.* If a dependent variable Y is a function of an independent variable function is expressed as $y = f(x)$ or $y = F(x)$ and read as Y is a function of x.

If the value of a set A belongs to x, generate another set B consists of values of y, then the function is expressed as f: A'!B and read as f is a function from A into B or A maps into B.

Here set A is called the domain of the function and set B is called the range of the function. For any orbitrarity value "a" belongs to x, gives a unique value f(a) corresponding to y.

Generally for n independent variables x_1, x_2, x_3 .... x_n, and y is a dependent variable the function can be given as $y = f(x_1, x_2, x_3 \ldots\ldots\ldots\ldots\ldots, x_n)$

Some examples of functions are given by

$y = x - 15 \qquad\qquad 0 < x < 1$

$y = x^2 + 6x - 2 \qquad\qquad 1 < x < 2$

In (i), the domain of x is the closed interval [0, 1] and the range of y is [-15, -14]. In the (ii), the domain of x is the interval [1, 2] and the range of y is [5, 14].

1.7 Construction of Functions

Many times in business, we commonly talk about profit functions, loss functions, cost functions and revenue functions, production function, demand and supply function, consumption function, into function, onto function, polynomial function, Absolute value function, Step function, Inverse function, Rational and Irrational function, Algebraic function. The functions are usually set up following the definition and calculation of the functional values. Now we will take some few examples to illustrate the method of constructing such fun-ctions.

Example 1: *A factory has 100 items on hand for shipment to a destination at the cost of Rs 1 a piece to meet a certain demand d. In case the demand d overshoots the supply. It is necessary to meet the unsatisfied demand by purchases on the local market at Rs 2 a piece. Construct the cost function if x is the number shipped from the factory.*

Solution: Suppose C(x) denotes the cost function.

If $d \geq x$,

$$C(x) = x + 2(d-x)$$

If $d < x$, $\quad C(x) = x$

It can be combined into a single representation by writing

$$C(x) = x + 2\max(0, d-x)$$

This definition of C(x) is seen to be equivalent to the earlier one by considering separately the cases $d \geq x$ and $d < x$.

Example 2: *If $f(x) = 2x + 1$, find the range if domain is {-1,2,3} and hence find the function.*

Solution: Here $f(x) = 2x + 1$

$= f(-1) = -1 \times 2 + 1 = -1,$

$= f(2) = 2 \times 2 + 1 = 5,$

$= f(3) = 3 \times 2 + 1 = 7$

So, Range = {-1, 5, 7}

And $f = \{(-1, -1), (2, 5), (3, 7)\}$

1.8 Classification of Functions

We have already learnt that for a function $f : A \rightarrow B$, f associates all elements of set A to set B and each element of set A is associated to a unique element of a set B. Thus, we may associate different element of set B or we may associate more than one element of set A to same element of set B (but same element of set A cannot be associated with more than one element of set B). Also, all elements of set B may or may not have their pre-images in A.

1.8.1 Functions Related To Economics:

In the case of functions in economics, the variables are hypothetical quantities and not actual observable quantities as in physical science.

The range and the domain of economics functions are made up of nonnegative quantities so that the graphs of these functions are in first quadrant only. In other words, for the purpose of economic analysis, only that part of a curve is relevant which lies in the first quadrant.

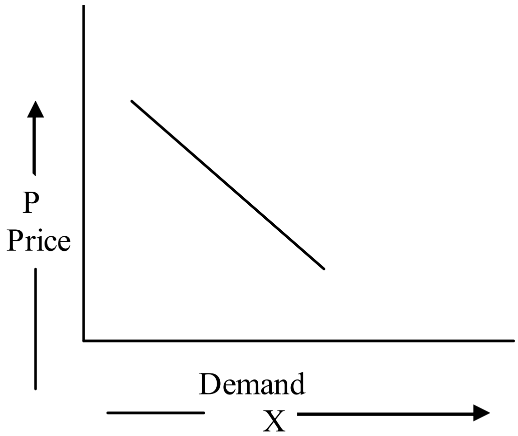
1. Demand Function:

As we know that the quantity demanded of a particular commodity by the buyers in the market is depending on the price. As the prices increases, the demand is decreases shown as figure. If q is the quantity of a commodity demanded and p is the price then the demand function is given by

$q = f(p)$ shows q depends on p price

$p = g(q)$ shows p depends on q

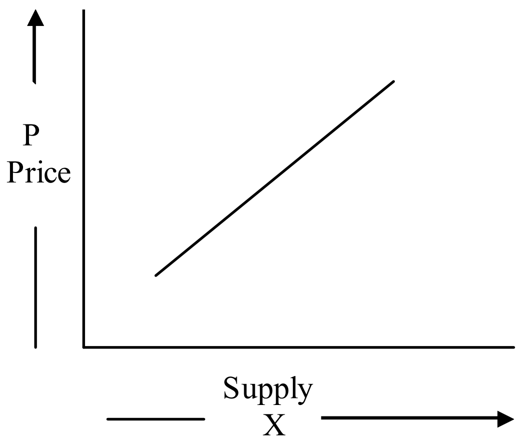
for example: $q = a - bp$ or $q = (10/p)$ or $q = -10p^2 + p + 5$



Demand Function

2. Supply function:

Price of any particular commodity in the market depends on the quantity of supply. As the quantity of supply increases the price is also increases. If x is the quantity of supply and p is its price then the supply function is given by $x = f(p)$ or $p = f(x)$.



Supply Function

3. Cost Function:

3.1. Total Cost Function **3. 2.** Average Cost Function

3.1 Total cost function: The total cost (C) which is equal to sum of fixed cost and variable cost, of production in a firm, is depending on the quantity produced of a particular commodity. As the production increases the total cost also increases. If x is the quantity produced at total cost T then the total cost function is given by $T = f(x)$.

3.2 Average Cost Function: Average cost function or cost per unit is obtained by dividing the total cost function by the quantity produced. Average cost function A = (T / x) = (*f* (x) / x).

4. Total Revenue Function:

The total revenue in a particular commodity of a firm is depending on the number units sold. If p is the price of any commodity and q is the quantity sold then the total revenue function is given by R = p.q, and the average revenue function or revenue per unit is obtained by dividing the total revenue function by the units of quantity sold, it is equal to price (p) of the commodity.

$AR = (R / q) = (Pq / q) = p = price$

5. Profit Function:

Profit on a particular commodity of a firm is equal to the difference or total revenue and the total cost the profit function is given by z = R – T.

6. Production Function:

The output product is depending in the various inputs like capital, labour, raw material etc. If q is the output quantity of a particular commodity and x, y and z are the input variables in a firm then the production function as sin y.

$q = f(x, y, z)$

For example: If labour (1) and capital (K) are inputs variables then the output is given by $Q = f(L, K) = AL^{\alpha} - K^{\beta}$, $\alpha + \beta = 1$ where A, α and β are constants is known as Cobb-Douglas production function.

7. Consumption Function:

The total consumption function of a firm is depending on the income as income increases the consumption expenditure also increases. If C is the consumption and I is the income then the consumption function given by

$C = f(5) = a + bI$ where a, b are constants.

Example1: *A company sells x tins of chicken each day at Rs. 80 per tin. the cost of production and selling price of these tins is Rs. 50 per tin plus a fixed daily overhead cost of Rs. 18,000. Determine the profit function. What is the profit if 2,000 tins are produced and sold a day? Find out the number of tins produced in a day with no profit and no loss.*

Solution: a). Per day x tins of chicken are produced with cost Rs. 80 per tin the revenue received per day is Revenue function

$R(x) = 80.\ x$

The cost of production per day cost function is $c(x) = 1{,}800 + 50x$

If P (x) is the profit function is given by

$P(x) = R(x) - c(x)$

$= 80x - (50x + 1{,}800) = 30x - 1{,}800$

If 2,000 tins are produced and sold in a day then the profit is given by

$P(2{,}000) = 30 \times 2{,}000 - 18{,}000$

$= 60{,}000 - 18{,}000 = Rs.\ 42{,}000$

b). for no profit and no loss then profit function is zero

$P(x) = 0 = 30x - 18{,}000 = 0$

$30x = 18{,}000x = (18{,}000 / 30) = 600$

600 tins per day.

1.8.2 Some Special Functions:

In this case, we have some special functions which can be used in economics as well as computer science area too. These functions are defined as follows:

1. Polynomial function:

The general form of the function

$f(x)$ or $y = f(x_1, x_2, x_3, \ldots, x_n) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0$ ……(i)

Where a_i's are real numbers, $a_1 \neq 0$, and n is positive integer is called a polynomial of degree n. The real constants $a_n, a_{n-1}, a_{n-2}, a_0$ are called *coefficients*. The exponent n is called the degree of the polynomial. For n = 1, the equation (a) can be written as

$y = a_1 x + a_0 \quad (a \neq 0)$

This is usually written as

$y = a + bx \quad (b \neq 0)$

Where $a = a_0$ and $b = a_1$. Such Polynomial functions are called *linear function*

(i. e. degree 1) and has the domain $\{x: x \in R\}$.

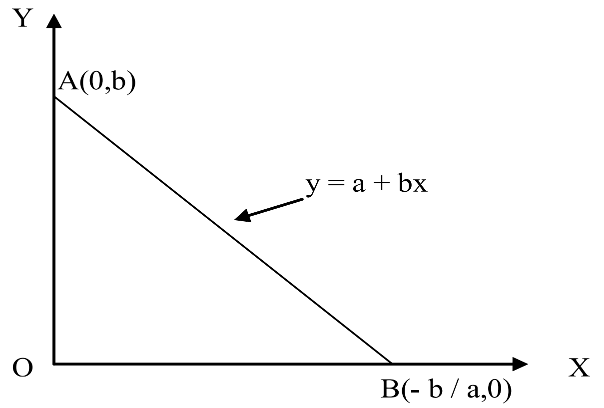
(b) Now for n = 2, the equation (i) can be written as:

$y = a_2 x^2 + a_1 x^1 + a_0 \quad (a_2 \neq 0)$ or $y = ax^2 + bx + c, \quad (a \neq 0)$

Where $a = a_2$, $b = a_1$ and $c = a_0$. Such polynomial functions are called second degree functions or *quadratic functions.*

The x -intercept of a quadratic function is very important. The x- intercept is that point (or points) at which the graphs crosses the x-axis; that is, the point at which y = 0. If such points exist, they satisfy the equation: $ax^2 + bx + c = 0$.

The graph of the linear function is the straight line y = a + bx which cuts the x-axis at a distance -b/a units from the origin and y-axis at a distance of a units from the origin as shown.



Inverse Mapping

Functions like $f(x) = x^3 + \sqrt{2x}$

$g(x) = 2 + x - x^4$

are polynomial function but $2x + x^{2/3}$ is not a polynomial function.

Example 1: *The demand for certain item is given by q = 150 - 3p Where q denotes the amount demanded and p the price per unit. It costs Rs 4 to produce each unit. What is the profit function of the firm for this item?*

Assume Y = total profit

R = total revenue

C = total cost

Since R = pq, C = 4q,

Profit = Total Revenue – Total cost

Y= R - C

= pq - 4q

= q (p - 4)

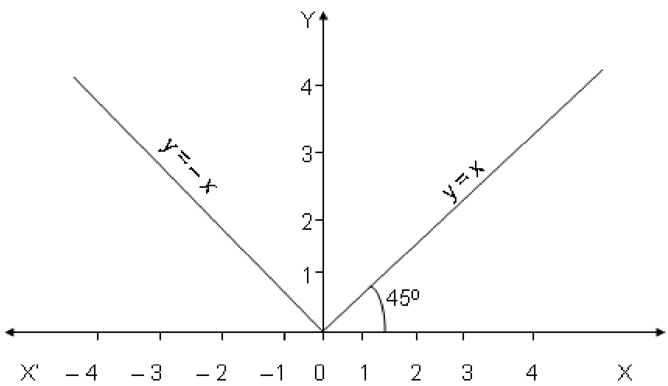
= (150 – 3p) (p – 4) [Substituting q = (150 – 3p) given]

= $-3p^2 + 162p - 600$ which is a quadratic function in p.

2. Absolute Value Function:

The functional relationship of the form *f*(x) or y = |x| is known as an *absolute value function,* where x represents the absolute value of x and is defined as given below, and the graph of this function is also shown

$$y = |x| = \begin{cases} x, & \text{for } x = 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

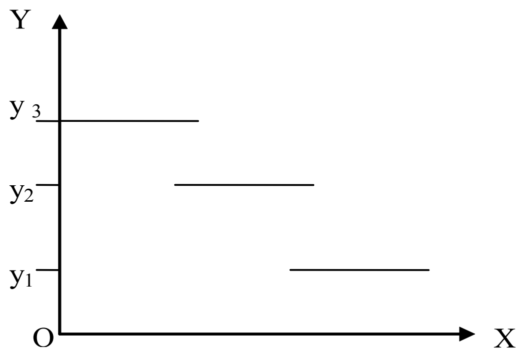


3. Step Function:

If a function is defined on a closed interval [a, b] and assume a constant value in the interior of each sub-interval say $[a, x_1]$, $[x_2, x_3]$, …………. , $[x_n, b]$ of [a, b] where $a < x_1 < x_2 < \ldots\ldots < x_n < b$, then such function is called a *step function.* Symbolically it may be expressed as:

y or *f* (x) = k_i for all values of x in the *ith* sub-interval.

The graph of this function is given below:



Step Function

4. Convex Set and Convex Function:

A set S of points in the two-dimensional plane is said to be convex if for any two points (x_1, y_1) and (x_2, y_2) in the set the line segment joining these points is also in the set.

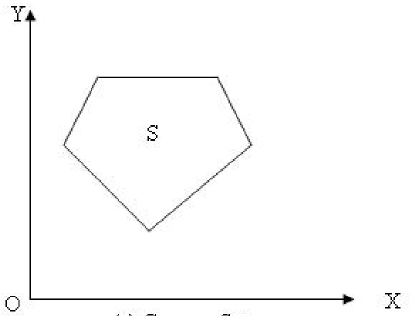
Mathematically, this definition implies (x_1, y_1) and (x_2, y_2) are two different points in S, Then the point whose coordinates given by

$\{\lambda x_1 + (1-\lambda) x_2; \lambda y_1 + (1 -\lambda) y_2\}$; $0 \leq \lambda \leq 1$

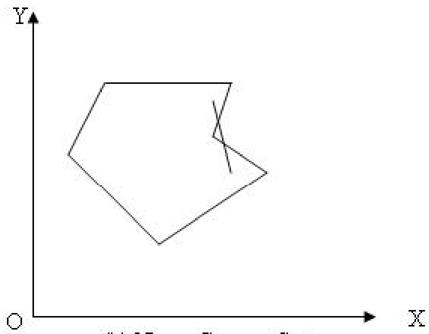
must also be in the set S.

If $\lambda = 0$, then we get the coordinates (x_2, y_2) n the given point. But, if $\lambda = ½$, the corresponding point on the line segment is $[(x_1+x_2)/2, (y_1+y_2)/2]$. This point lies in the centre of the line segment joining two points(x_1, y_1) and (x_2, y_2).

The examples of convex set are circle and triangle. The examples are given below for convex and non convex sets.



(a) Convex Set



(b) Non - Convex Set

Convex function: A function f(x) defined over a convex set S is said to be *convex function* if for any two distinct points x_1 and x_2 lying in S for any $0 \leq \lambda \leq 1$, $f\{\lambda x_1 + (1-\lambda) x_2\} \leq \lambda f(x_1) + (1-\lambda) f(x_2)$.

5. Inverse Function:

If variable x and y are interdependent such that (i) y is the function of x; $y = f(x)$

(ii) If x is the function of y, $x = g(y)$, then f is termed as the inverse of g and vice-versa. For example, if $y = x^2 + 2x + 6$, then the inverse function is: $x = -1 \pm \sqrt{y-6}$

These functions are related to as follows:

$$y = f(x) = f[g(y)] \quad \text{or} \quad fog(y)$$

and $$x = g(y) = g[f(x)] \quad \text{or} \quad gof(x)$$

Consequently $fog = gof$. Functions fog and gof are termed as *composite functions.*

6. Rational functions:

A rational function is defined as the quotient of two polynomial functions and is of the form:

$$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots\ldots\ldots + b_0} = \frac{P(x)}{Q(x)}, \quad Q(x) \neq 0$$

Where P (x) is a polynomial function of degree n and Q (x) is a non zero polynomial function of degree m.

An expression which involves root extraction on terms involving x is called *irrational function. The function such as* $\sqrt{\;}x$, $\sqrt{2x^2+7x+6}$ are examples of irrational functions.

7. Algebraic Function:

A function consisting a finite number of terms involving powers and roots of the variable x and the four basic mathematical operations (addition, subtraction, multiplication and division) is called an algebraic function. In general, it can be expressed as

$y^n + A_1 y^{n-1} + \ldots\ldots\ldots\ldots + A_n = 0$ where $A_1, A_2, \ldots\ldots\ldots, A_n$ are rational functions of x.

There are two categories of algebraic functions namely: explicit and implicit algebraic functions. For example, y = $\sqrt{x+2x^3}$ is and enplicat algebraic function, where as $xy^2 + xy + x^2 = 0$ is an implicit function.

8. Transcendental Function:

All functions which are not algebraic are called transcendental functions. These functions include:

1. Trigonometric Functions:

The trigonometric functions of an angle θ (θ be any real number) are given by:

$\sin\theta = \sin\theta^c$, $\cos\theta = \cos\theta^c$, $\tan\theta = \tan\theta^c$
$\operatorname{cosec}\theta = \operatorname{cosec}\theta^c$, $\sec\theta = \sec\theta^c$, $\cot\theta = \cot\theta^c$
where θ denote the angle whose radian measure is in θ.

The sin and cosec are said to be co - functions, as are the cos and sec and tan and cot.

The trigonometry functions are defined similarly for negative angles. An angle θ may be measured in degrees or radians. However, in calculus and its applications to business and economics radian measure is usually more convenient.

The trigonometric functions are very useful in the study of business cycles, seasonal or other cyclic variations are described by sine or cosine functions.

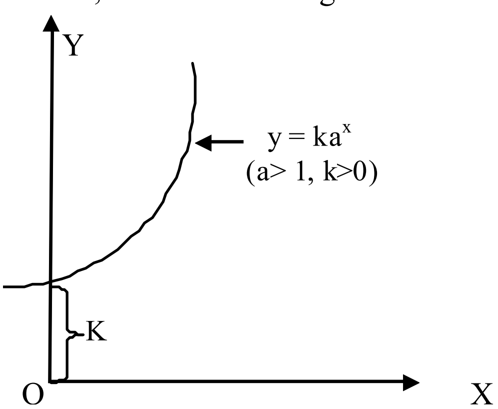
2. Exponential Functions:

A function having a constant base and a variable exponent is called an exponent function, such as

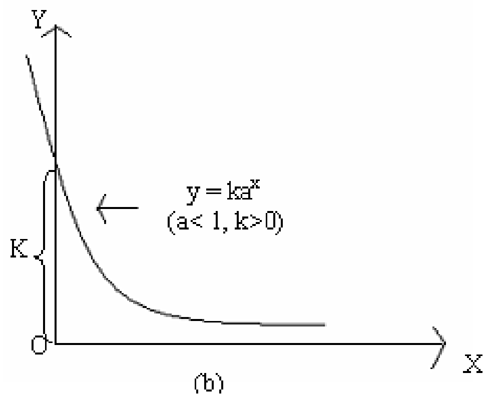
(i) $y = a^x$, $a \neq 1$, $a > 0$ (ii) $y = k\,a^x$ $a \neq 1$, $a > 0$
(iii) $y = k\,a^{bx}$ $a \neq 1$, $a > 0$ (iv) $y = k\,e^x$

where a, b, e, and k are constants and x is exponent.

In calculus and its applications to business and economics, such functions are useful for describing sharp increase and decrease in the value of dependent variable. For example the graph of exponential function $y = ka^x$ indicates rise to the right in the value of y for a > 1 and k > 0 whereas indicates fall to the left for a < 1 and k >0, as shown in the fig.



(a)



(b)

The rules governing the exponents are as under:

(i) $a^{x_1}.a^{x_2} = a^{x_1+x_2}$ (ii) $a^{x_1}/a^{x_2} = a^{x_1-x_2}$ (iii) $(a^{x_1})^{x_2} = a^{x_1 x_2}$

(iv) $(a.b)^{x_1} = a^{x_1}.b^{x_1}$ (v) $(a/b)^{x_1} = a^{x_1}.b^{-x_1}$ (vi) $a^0 = 1$.

3. Logarithmic Functions:

A logarithmic function is expressed as $\log_a x$ where $a > 0$, $a \neq 1$ is the base. It is read as "y is the log to the base a of x". This relationship may also be expressed by the equation $x = a^y$. It is an exponential function. Thus, logarithmic and exponential functions are inverse functions, i.e. if x is an exponential function of y, then y is a logarithmic function of x.

Although the base of logarithm can be any positive number other than 1, but most widely used bases are either 10 (common or Briggsian logarithms) or e = 2.718 (natural or Naperian logarithms). By convention, log x denotes the common logarithm of x in x denotes natural logarithm of x. If any other base is meant, it is specified.

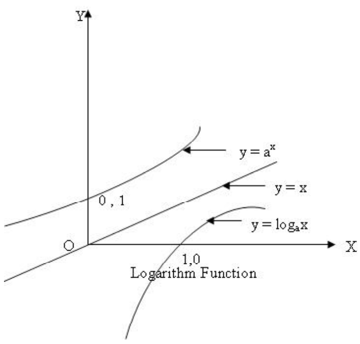
Some important properties of the logarithms are as follows. If x and y are positive real number, then

(i) $\log_a xy = \log_a x + \log_a y$ (ii) $\log_a (xy) = \log_a x - \log_a y$ (iii) $\log_a x^n = n \log_a x$

(iv) $\log_a x^{1/n} = (1/n) \log_a x$ (v) $\log_a x = \log_a b \times \log_b x$

(vi) Logarithm of zero and negative number is not defined.

Since exponential function $x = a^y$ and logarithmic function $y = \log_a x$ are universe functions, therefore graph of these curves for a particular value of 'a' can be obtained from the graph of each other by taking reflection about the line y = x, as shown in the fig



Logarithm Function

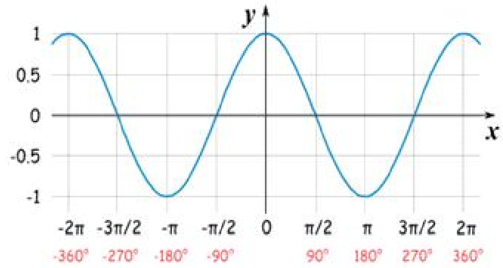
Incommensurable Power Functions: A function having a variable base and a constant exponent is called and incommensurable power function, such as y = $x^{\sqrt{5}}$ or $x^{3/2}$ or x^a etc.

9. Even and Odd Functions:

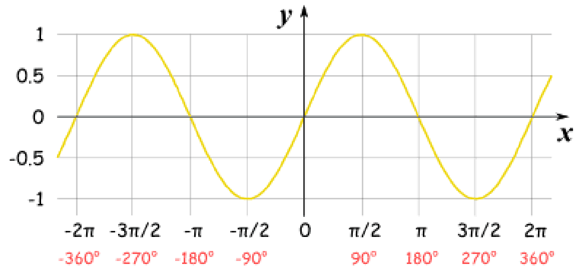
If a function not change its sign of its independent variable changed, then it is said to be an even function, i.e. f(-x) = f(x). The examples of even functions are x^6, cos x, etc. It follows that the graph of an even function is symmetrical about the y - axis.

On the other hand, f(x) is said to be odd if f(-x) = -f(x). The examples of odd functions are: x^7, 10. Periodic Function: If f(x + T) = f(x), where T is a real number, then f(x) is called a periodic function. The real number T is called a period of f(x).

The least positive period of a periodic function is called the principal period of that function. Since for all real numbers x, sin(2π + x) = sin x and cos (2π + x) = cos x, therefore the function sin x and cos x are periodic functions with period 2π. Other than this 4π, -2π, 6π etc. are also periods of sin x and cos x.



Cosine Function



Sine Function

1.9 Roots of Functions

The value (or values) of x at which the given function f(x) becomes equal to zero are called zeros of the function f(x). The zeros of the function are also called the roots of the given function f(x).

For the linear function $y = ax + b$ the roots are given by $ax + b = 0$, i. e. $x = - b/a$.

Thus, if $x = - b/a$, then substituting in the given equation, $ax + b = 0$, the left hand side of its becomes equal to its right hand side. For the quadratic function $y = ax^2 + bx + c$, $a \neq 0$ the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ has to be solved to find the roots of the function y. The general value of x which satisfy the given quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This shows that, in general, there are two values of x (also called roots) for which

$ax^2 + bx + c$ become zero. The two values are

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Note: The numbers of roots of the given function are always equal to the highest power of the independent variable.

1.10 Break-Even Analysis

Break Analysis is often referred as cost – volume – profit analysis. This is because the purpose of the analysis is to study the relationship among cost, volume and profit in order to determine the point of break-even volume. The major components of the break even model are:

(i) Total Revenue: It consists of selling price (p) multiplied by the quantity sold (q) i.e., $R = p.q$

(ii) Fixed Cost: The cost which remains constant (in rupees) regardless of the volume of production. Let it be Rs. k.

(iii) Variable Costs: The costs which vary with the quantity produced or sales. It is calculated by multiplying cost/unit of the output with the volume of production, i.e., $V = v.q$, where v is the variable cost/unit of output.

The break - even analysis seeks to determine the output at which the producers will break-even, i.e. the level of output where his costs will exactly meet his revenue.

The total profit associated with an output of q units is given by

Profit (P) = Total revenue – Total cost

= (Price) (Quantity) – {Fixed cost + (Variable cost) (Quantity)}

$= p \cdot q - (k + v \cdot q)$

By definition, break- even analysis determines the optimum value of q for which profit P equals zero, i.e. Total revenue = Total cost

or $p \cdot q - (k + v \cdot q) = 0$

$$q^* \text{ (optimum)} \quad \frac{k}{p - v} = \frac{\text{Fixed Cost}}{\text{Selling Price} - \text{Variable Cost}}$$

Solved Examples

Example 1. *A firm produces an item whose production cost function is C = 80 + 4x, where x is the number of items produced. If entire stock is sold at the rate of Rs. 8 then determine the revenue function. Also obtain the 'break – even' point.*

Solution. The revenue function is given by R = 8x. Also given that , C = 80 + 4x. Therefore

Profit, P = R – C = 8x – (80 + 4x) = 4x – 80

The break- even points occur when R – C = 0 or R = C, i.e. 8x = 80 + 4x or x = 20 (units).

Example 1. *A company producing dry cells introduces production bonus for its employees which increases the cost of production. The daily cost of production C(x) for x number of cells is Rs. (3.5x + 12,000).*

(a) If each cell is sold for Rs. 6 , determine the number of cells that should be produced to ensure no loss.

(b) If the selling price is increased by 50 paise, what would be the break – even point?

(c) If at least 6000 cells can be sold daily, what price the company should charge per cell to guarantee no loss?

Solution. Let R(x) be the revenue due to the sales of x number of cells.

(a) Given that, cost of each cell is Rs. 6. Then R(x) = 6x. For no loss, we must have

R(x) = C(x) or 6x = 3.5x + 12,000 or 12,000/2.5 = 4,800 cells.

(b) Increased selling price is, Rs.(6 + 0.50) = Rs. 6.5. Thus, R(x) = 6.5x. Now, for break- even point, we must have

R(x) = C(x) or 6.5x = 3.5x + 12,000 or x = 12,000/3 = 4000 cells.

(c) Let p be the unit selling price. Then revenue from the sale of 6000 cells will be, R(p) = 6000 p.

Thus, for no loss we must have

R(p) = C(p) or 6000p = 3.5 × 6000 + 12,000 or p = 33,000/6,000 = Rs. 5.5.

1.11 Sequence and Series (Progression)

1.11.1 Sequences

A sequence is a function, whose domain is the set N of natural numbers or rule set of N. We may also define a sequence as an *arrangement of numbers according to a certain fixed rule*. The different numbers in sequences are called its **terms** and are generally denoted by $a_{1,}\, a_{2,}\, a_{3,\ldots\ldots}$ or $T_{1,}\, T_{2,} T_{3\ldots\ldots}$ etc. Some examples of sequences are:

1, 3, 5, 7, 9, …… …(i)

1, 3, 9, 27, ….. …(ii)

In (i), the terms are arranged such that the difference between two consecutive terms is the same i.e. , 2.

In (ii), the terms are arranged in such a way that the ratio of two consecutive terms is the same i.e., 3.

Note. *It is obvious that if the rule (which the terms of a sequence obey) is known, then all or any term can be found.*

General Term

*The **nth term** of a sequence is called the general term and is denoted by $\boldsymbol{a_n}$ or $\boldsymbol{T_n}$.*

Finite and Infinite Sequences

Finite Sequence: *A sequence containing **finite** number of terms is called a finite sequence.*

For example, 2, 4, 6, 8, …, 40

1, 3, 9, …, 243

24, 21, 18, … upto 6 terms.

Infinite Sequence: *A sequence is called **infinite** if it is not a finite sequence or in other words, a sequence having infinite terms is called an infinite sequence.*

For example, 1, 3, 5, 7, …, to infinity

3, 33,333, …

Note. *If the general term (=$\boldsymbol{T_n}$) of a sequence is given, we can find $T_{1,}\, T_{2,} T_{3}$.... by putting n = 1, 2, 3, ... in $\boldsymbol{T_n}$.*

1.11.2 Series

The terms of a sequence connected by +ve or - ve sign from a series.

For example, 1+ 3+ 5+ 7+ …

The different terms of the series are denoted by or $T_{1,}\, T_{2,} T_{3\ldots\ldots}$

Solved Examples

Example 1. *Write the first three terms of the sequences whose nth terms are*

*(*i) 3n - 5 (ii) 2^n (iii) n^2

Solution. (i) Here $T_n = 3n - 5$

Putting n = 1, 2, 3 in T_n, we get

$T_1 = 3.1 - 5 = - 2$ $T_2 = 3.2 - 5 = 1$ $T_3 = 3.3 - 5 = 4$

(ii) Here $T_n = 2^n$

Putting n = 1, 2, 3 in T_n, we get

$T_1 = 2^1 = 2$ $T_2 = 2^2 = 4$ $T_3 = 2^3 = 8$

(iii) Here $T_n = n^2$

Putting n = 1, 2, 3 in T_n, we get

$T_1 = 1^2 = 1$ $T_2 = 2^2 = 4$ $T_3 = 3^2 = 9$

Example2. *If $T_n = n^2$, when n is odd and $T_n = 2n$, when n is even, then write down the first three terms.*

Solution. Here $T_n = n^2$, when n is odd

Putting n = 1, 3, 5 , we get

$T_1 = 1^2 = 1$ $T_3 = 3^2 = 9$ $T_5 = 5^2 = 25$

Also, $T_n = 2n$, when n is even

Putting n = 2, 4, 6 , we get

$T_2 = 2.2 = 4$ $T_4 = 2.4 = 8$ $T_6 = 2.6 = 12$

Hence, the first 6 terms of the sequence are 1, 4, 9, 8, 25, 12.

Example3. *If $T_n = (n - 1) (n - 2) (n - 3)$; show that the first three terms of the sequence are zero, but the rest of the terms are positive.*

Solution. Here $T_n = (n - 1) (n - 2) (n - 3)$

Putting n = 1, 2, 3 in T_n successively, we get

$T_1 = (1 - 1) (1 - 2) (1 - 3) = 0, T_2 = (2 - 1) (2 - 2) (2 - 3) = 0, T_3 = (3 - 1) (3 - 2) (3 - 3) = 0$

For n>3, each of the terms (n - 1) (n - 2) and (n - 3) is positive, and hence their product (n - 1) (n - 2) (n - 3) is also positive.

Hence, the first three terms are zero, but the rest of the terms are positive.

1.12 Arithmetic Progression

An arithmetic progression is a *sequence* whose terms increase or decrease by a constant number called the common difference. A series in arithmetic progression thus becomes an additive series in which the common difference can be found by subtracting each term from its preceding one. Thus,

(i) The sequence 1, 5, 9, 13, 17, 21, 25 … is an infinite arithmetic progression of seven terms, the first term is 1 and the common difference is 4. Similarly,

(ii) The sequence 12, 5,- 2, - 9, … is an infinite arithmetic progression of four terms , the first term is 12 and the common difference is 7.

The corresponding arithmetic series are

(i) 1 + 5 + 9 + 13 + 17 + 21 + 25 + …

(ii) 11 + 17 + 23 + 29 + ….

Thus if the first term and common difference are known, the A.P. is completely known,

The arithmetic progression a, (a + d), (a + 2d), (a + 3d) …

Whose first term is a and the common difference is d, is designated *as the standard form of the arithmetic progression.*

The corresponding arithmetic series a + (a + d) + (a + 2d) + (a + 3d) +…

is designated *as the standard form of an arithmetic series.* The abbreviation ' A. P.' for arithmetic progression is commonly used. In other words, a sequence in which $T_n - T_{n-1} =$ constant difference (say d), is called an A. P.

This difference is called the common *difference* and is denoted by d.

Note. 1. In general, a, a + d, a + 2d … represents an A.P. whose first term is T_1 and common difference is d.

2. nth term (T_1) of an A. P. is called its ***general term.***

***n*th term of an A. P.**

Let the A. P. be an a, (a + d), (a + 2d), (a + 3d), … whose first term = a and common difference = d Then,

$$T_1 = a = a + (1 - 1)\, d = a + 0.\, d$$

$$T_2 = a + d = a + (2 - 1)\, d$$

$T_3 = a + 2d = a + (3 - 1)\, d =$ and so on. So, nth term of an A. P. is

$$T_n = a + (n - 1)d$$

Properties of an A.P.

If each of the term of an A. P. is

(i) increased by (ii) decreased by (iii) multiplied by (iv) divided by a constant quantity, then the resulting sequence is still in A.P.

Let the A.P. be a, (a + d), (a + 2d), ….. and k is a constant quantity then, if we add, subtract, multiplied or divide the series by k then it will still remain an A.P.

Example 1. *Find the 12^{th} term of the A. P. 3, 7, 44, 15…*

Solution. Here a = 3, d = 7 - 3 = 4

$$\text{So, } T_n = a + (n - 1)\, d$$

$$T_{12} = 3 + (12 - 1)4$$

$$= 3 + 44 = 47$$

Example 2. *Which term of series 12 + 9 + 6 + ... is equal to (i) - 30, (ii) - 100?*

Solution. (i) The series in an A.P. with first term 12 and common difference *-3.*

$T_n = a + (n - 1)\, d = 12 + (n - 1)(- 3) = 15 - 3n$

Suppose the nth term is - 30 then $15 - 3n = - 30 \rightarrow n = 15$

So, - 30 is the 15^{th} term.

(ii) Now suppose that $T_n = - 100$, $15 - 3n = - 100 \rightarrow n = \frac{115}{3}$

Which is impossible, because n must be a whole number. Hence there exists no term in the series which is equal to - 100.

Sum of the first n terms of the general A. P.:

The sum of a series in A. P. is an important quantity which yields many other related results. We denote the sum of n terms by S_n and the first and last terms of the sequence by *a* and *l* respectively. Let S_n denote the sum of n terms, then

$S_n = a + (a + d) + (a + 2d) + \ldots\ldots + (l - 2d) + (l - d) + l$ .... *(1)*

Writing the above series in reverse order, we have

$S_n = l + (l - d) + (l - 2d) + \ldots\ldots + (a + 2d) + (a + d) + a$ … (2)

Adding (1) and (2), we get

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots\ldots\ldots \text{ up to n terms} = n (a + l)$$

$$Sn = \frac{n(a+1)}{2}$$

But $l = a + (n - 1) d$

$$Sn = \frac{n[a+a+(n-1)d]}{2}$$

Note: *Generally, formula* **(I)** *is used when the last term is given and* **(II)** *is used when Common difference is given.*

Example 1. *Find the sum of the series: 72 + 70 + 68 + ...+ 40*

Solution. The given series is an A. P. with a = 72 and d = -2. Let n be the number of terms. Then $T_n = 40$.

$a + (n - 1) d = 40 \rightarrow 72 + (n - 1)(-2) = 40$, i. e., 2n = 34 or n = 17.

Now

$$Sn = \frac{n(a+1)}{2}$$

Example 2. *The first and the last term of an A. P are respectively - 4 and 146, and the sum of the A.P. = 7171. Find the number of terms of the A.P. and also its common difference.*

Solution. We have a = - 4 and *l* = 146 and $S_n = 7171$

Let n be the number of terms of the A.P.

$$Sn = \frac{n(a+1)}{2} \rightarrow 7171 \ \ or, Sn = \frac{n(-4+146)}{2} \rightarrow 7171$$

n = 101, Also $146 = (-4) + (101 - 1) d \rightarrow 100d = 146 + 4 = 150$

d = 1.5. Hence n = 101 and d = 1.5

Example 3. *Find the sum of all numbers between 100 and 1000. Which are divisible by 11?*

Solution. Numbers between 100 and 1000 divisible by 11 are 110, 121, 132… 990, which clearly form an A.P., where first term, a = 110; the common difference d = 11, and the *n*th term T_n = 990. Then, we have $T_n = a + (n - 1) d$

$$990 = 110 (n - 1) 11, \text{ i.e., } n = 81.$$

So the sum of these numbers is given by

$$S_n = \frac{n(a + l)}{2} \qquad S_n = \frac{81(110 + 990)}{2} = 44550$$

Example 4. *Find the series whose sum to n terms is 5* n^2 *+ 3n.*

Solution. Given that $S_n = 5n^2 + 3n$. Pitting n = 1, and 2, we get

$$S_1 = 5(1)^2 + 3(1) = 8 \text{ and } S_2 = 5(2)^2 + 3(2) = 26$$

Now first term, $a = S_1 = 8$, second term $S_2 - S_1 = 26 - 8 = 18$. Then the common difference, d = second term - first term = 18 - 8 = 10.Hence, the required A.P. is: 8, 18, 28, 38 …

Some Important Formulae

(a) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

(b) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{2}$

(c) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n(n+1)^2}{2}$

Arithmetic Mean (A.M.)

If a, A, b are in A.P., then A is the A.M. between a and b, i.e., if three quantities are in A.P., then the middle one is defined as A.M. between the other two.

Arithmetic Mean Between 'a' and 'b'

Let A be the A.M. between a and b.

So, a, A, b are in A.P.

Thus, $A - a = b - A$, i.e., $2A = a + b$

Similarly, $A = \dfrac{a+b}{2}$

To insert *n* A.M's between two quantities 'a' and 'b'

Let $A_1, A_2, A_3, \ldots, A_n$ be the *n* A.M's between a and b.

So, a, A_1, A2, $A_3 \ldots A_n$, b are in A.P.

In this A.P., the first term is a, the last term is b and the number of terms is (n + 2) i.e., b is the (n + 2)th term= a + (n + 2 - 1)d = a + (n + 1)d

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + 2.\frac{b-a}{n+1}$$

…………………………

$$A_n = a + nd = a + n.\frac{b-a}{n+1},$$

Which are the required arithmetic means between a and b.

Example 1. *The first and the eight terms of an A.P. series are 2 and 23 respectively. Find the tenth term of the series.*

Solution. Let common difference = d

$T_8 = 2 + (8 - 1)d \rightarrow 23 = 2 + 7d \rightarrow d = 3$, Tenth term $T_{10} = 2 + (10 - 1)3 = 29$

Example 2. *The pth, qth, and rth terms of an A.P. series are a, b, and c respectively. Prove that, (q - r)a + (r - p)b + (p - q)c = 0*

Solution. Let the first term = a, and the common difference = d

According to the problem we have.

$$a = a_1 + (p - 1)\, d \qquad \dots (1)$$

$$b = a_1 + (q - 1)\, d \qquad \dots (2)$$

$$c = a_1 + (r - 1)\, d \qquad \dots (3)$$

Now, $(2) - (3) \rightarrow b - c = (q - r)\, d \qquad \dots (4)$

$(3) - (1) \rightarrow c - a = (r - p)\, d \qquad \dots (5)$

$(1) - (2) \rightarrow a - b = (p - q)\, d \qquad \dots (6)$

Now, Multiplying (4), (5), and (6) by a, b, and c respectively and then adding we get,

$$(b - c)a + (c - a)b + (a - b)c = \{(q - r)a + (r - p)b + (p - q)c\}d$$

$$\rightarrow \quad 0 = \{(q - r)\, a + (r - p)\, b + (p - q)\, c\}\, d$$

$$\rightarrow \quad (q - r)\, a + (r - p)\, b + (p - q)\, c = 0$$

Note. The relation to be shown contains (q - r), (r - p), and (p - q). If we subtract (3) from (2), (1) from (3), and (2) from (1) then and only then we will get (q - r), (r - p), and (p - q).

Example 3. *How many terms of this series 27 + 24 + 21 + 18 + ... will add up to 126?*

Solution. Let the first n terms of this series add up to 126.

$S_n = 126$, $a = 27$, $d = -3$, $n = ?$

$$Sn = \frac{n[2a + (n-1)d]}{2}$$

$$Sn = \frac{n[54 + (n-1)d]}{2}$$

$$\rightarrow \quad 126 = \frac{n[54 + (n-1)(-3)]}{2}$$

$= \quad 252 = n\,[54 - 3n + 3]$

$= \quad 252 = n\,[57 - 3n]$

$= \quad 252 = 57n - 3n^2$

$= \quad n^2 - 19n + 84 = 0$

$= \quad n^2 - 7n - 12n + 84 = 0$

$= \quad (n - 7)(n - 12) = 0$

$= \quad n = 7 \text{ or } n = 12$

i.e., the sum of the first 7 terms is 126 and the sum of first 12 terms is also 126.

Example 4. *If the third and the sixth terms of an A.P. are 7 and 13 respectively. Find the first term and the common difference.*

Solution. Let the first term of the A.P. series is 'a' and its common difference is d.

Now, $T_n = a + (n - 1)\, d \rightarrow T_3 = a + (3 - 1)\, d \rightarrow 7 = a + 2d$

$T_n = a + (n - 1)\, d \rightarrow T6 = a + (6- 1)\, d \rightarrow 13 = a + 5d$

$(2) - (1) \qquad \rightarrow \quad 3d = 6 \rightarrow d = 2$

Now, putting the value of d in (1) we get a = 3, the 1st term = 3 and common difference = 2.

Example 5. *The sum of three numbers in A.P. is 33 and their product is 1287, Find the numbers.*

Solution. Let the three numbers be a - d, a, a + d. Clearly, these three numbers are in A.P.

According to the question a - d + a + a + d = 33

$\rightarrow 3a = 33 \rightarrow a = 11$

Again, $(a - d).\, a.\, (a + d) = 1287 \rightarrow a\,(a^2 - d^2) = 1287 \rightarrow 11(11^2 - d^2) = 1287 \rightarrow 121 - d^2 = 117$

$\rightarrow d^2 = 4 \rightarrow d = \pm 2$

When d = 2, then the three numbers will be: 11 - 2, 11, 11 + 2 i.e., 9, 11, 13

Again, when d = - 2, then the three numbers will be: 11 - (- 2), 11, 11 + (- 2) i.e., 13, 11, 9.

1.13 Geometric Progression

A geometric progression is a sequence whose terms increase or decrease by a constant ratio called the common ratio. A series in geometric progression thus is a multiplicative series whose common ratio can be found by dividing any term by its preceding term. Thus

(i) The sequence 1, 2, 4, 8, 16, 32 … is an infinite geometric progression, the first term is 1 and the common ratio is 2. Similarly,

(ii) The sequence 5, -10, 20, -40, 80 … is a geometric progression, the first term is 5 and the common ratio is -2.

The corresponding geometric series are:

1 + 2 + 4 + 8 + 16 + 32 + …

5 - 10 + 20 - 40 + 80 - …

The geometric progression is, therefore, in the form: a, ar, ar^2, ar^3, ar^4 …

Whose first term is a and the common ration is r, and is designated as the standard form of a progression. The corresponding geometric series is $a + ar + ar^2 + ar^3 + ar^4 + \ldots$ and is designated as the standard form of geometric series. The abbreviation commonly used for 'geometric progression' is G.P.

In other words, a sequence of non-zero numbers $a_1, a_2, a_{3\ldots}\ a_n \ldots$ is called a **geometric progression** if $\frac{a_k + 1}{a_k} = r$ (constant), for $k \geq 1$ and $k \in N$.

*n*th term of a G.P. is called its **General Term**.

General Term of a G.P.

Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Then,

$T_1 = a = ar^{1-1} \qquad T_2 = ar = ar^{2-1} \qquad T_3 = a\,r^2 = ar^{3-1} \ldots\ldots T_n = ar^{n-1}$

Thus the *n*th term of a G.P. is given by $\mathbf{T_n = ar^{n-1}}$

Remark. If the *n*th term is the last term and is denoted by '*l*', then $l = ar^{n-1.}$

Solved Examples

Example 1. *Which term of the G.P.? 5, 1, 0.2 ... is 0.0016*

Solution. Here a = 5, r = 1/5 $T_n = 0.0016 = 16/10000$

$16/10000 = 5. (1/5)^{n-1}$ i.e., $(1/5)^{n-1} = (1/625) = (1/5)^4$

n - 2 = 4 '! n = 6. Hence 6th term is 0.0016.

Example 2. Find the 10th term of the series: 4 - 8 + 16 - 32 +...

Solution. Given a = 4 and r = -2. Thus,

$$T_n = ar^{n-1} = T_{10} = ar^{10-1} = 4(-2)^9 = -2048.$$

Example3. Three numbers whose sum is 15 are in A.P. If 1, 4, and 19 are added to them respectively, the results are in G.P. Find the numbers.

Solution. Let the three numbers in A.P. be a - d, a, a + d, so that

a - d + a + a + d = 15 or a = 5

But, numbers (a – d) + 1, a + 4, (a + d) + 19 are in G.P., so that

$\{(a-d)+1\}\{(a+d)+19\} = (a+4)^2$ or $(5-d+1)(5+d+19) = (5+4)^2$

$$(6-d)(24+d) = 81 \text{ or } d^2 + 18d - 63 = 0$$

$$(d-3)(d+21) = 0, \text{ i.e., } d = 3 \text{ or } d = -2.$$

Hence, the required numbers are

5 – 3, 5, 5 + 3 or 5 – (– 21), 5, 5 + (– 21), i.e. 2, 5, 8 or 26, 5, – 16.

Example 4. *If the third term of G.P. is the cube of the first and the 6^{th} term are 64. Find the series.*

Solution. Let a be the first term and r be the common ratio. Then, the general term is given by

$$T_n = a\,r^{n-1}.$$

Therefore, $T_3 = a\,r^{3-1} = a\,r^2$ and $T_6 = a\,r^{6-1} = a\,r^5$

But, we are given that $T_3 = (T_1)^3$, i.e. $a\,r^2 = a^3$ or $r = a$

Also, $T_6 = 64$, or $a\,r^5 = 64$

or $a.\,a^5 = 64$, $a^6 = 64$ or $a = 2$

Thus, the first term is, a = 2 and common ratio r = 2. Hence, the series is 2 + 4 + 8 + ….

Sum of '*n*' terms of G.P.

Let a be the first term and r the common ratio of the given G.P. of *n* terms and its sum to *n* terms be denoted by S_n. Then,

$S_n = a + ar + a\,r^2 + a\,r^3 + \ldots\ldots. + a\,r^{n-2} + a\,r^{n-1}$ ….(1)

Multiplying both sides by r, we get

$r.\,S_n = ar + a\,r^2 + a\,r^3 + + a\,r^4 + \ldots\ldots. + a\,r^{n-1} + a\,r^n$ ….. (2)

Subtracting (2) from (1) we get,

$$(1 - r)\,S_n = a - a\,r^n = a\,(1 - r^n)$$

$$S_n = \frac{a\,(1 - r^n)}{1 - r}, \text{ for } r < 1 \qquad \text{... (3)}$$

$$S_n = \frac{a\,(r^n - 1)}{r - 1}, \text{ for } r > 1 \qquad \text{... (4)}$$

Also, $\mathbf{S_n}$ = a + a + a + a + ...to n terms = **na** for r = 1

Note: If *l* denotes the last term i.e. the *nth* term, then for r > 1.

$$S_n = \frac{a\,(r^n - 1)}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{ar^{n-1}.r - a}{r - 1} = \frac{lr - a}{r - 1} \qquad \text{.... (5)}$$

Also,

$$S_n = \frac{a - lr}{l - r} \quad \text{if } r < 1 \qquad \text{.... (6)}$$

Remarks

1. It is convenient to use formulae (3) and (6) when r < 1. and (4) and (5) when r > 1.

2. (a) If r = 1, then S_n = 0/0, which is meaningless. But in this case G.P. reduces to a, a, a ... and, therefore,

S_n = a + a + a + ... to *n* terms.

(b) If the number of terms in a G.P. are infinite, then

$$S_n = \begin{cases} \dfrac{a}{1 - r} & \text{when } r < 1 \\ \dfrac{a}{r - 1} & \text{when } r > 1 \end{cases}$$

Sum of an infinite G.P.

A series consisting of infinite (unlimited) number of terms is called an infinite series. In the determination of the sum of an infinite G.P., a, ar, a r^2, a r^3 ...... three cases may arise:

$S_{n=}$ a $\mathbf{(r^n - 1)}$ will become $S_{n=}$ “, for n = “

(r – 1)

Case 1. If r > 1, it follows that the value of each term of series is greater than its preceding term. Then

$$Sn = \frac{(r^n - 1)}{(r - 1)}\, will\ become\ S_n = \infty\ for\ \infty$$

Case 2. If r = 1, in this case each term of the series is 'a', then the sum S_n for $n = \infty$ is given by $S_\infty = a + a + a + \ldots\ldots$ to ∞

That is, in this case also the sum will be " whatever the value of 'a' (>0) may be.

Case 3. If r < 1, in this case the value of each term will decrease with increasing value of *n,* because $r^n \rightarrow 0$ (tends to zero) as n becomes infinitely large, i.e., r^∞ 0 when r < 1. We have

$$S_n = \frac{(\mathbf{a - ar^n})}{\mathbf{(1 - r)}} = \frac{a}{1-r} - \frac{ar^n}{1-r} \quad (r < 1)$$

When n becomes infinitely large, the quantity ar' will be very small to count.

Therefore, for sufficient large value of n (i.e., = ∞) we can neglect the value of $\frac{ar^n}{1-r}$

Thus the sum of the infinite number of terms in G.P. is given by $S_\infty = \frac{a}{1-r}$

Solved Examples

Example1. *Find the sum of the series: 1 + 5 + 25 + ... to 20 terms.*

Solution. Given a = 1, r =5 (> 1). Hence, using the formula

$$S_n = \frac{a\,(r^n - 1)}{r - 1} = S_{20} = \frac{1\,(5^{20} - 1)}{5 - 1} = 1/4\,(5^{20} - 1)$$

Example1. *Find the sum of infinite terms of the series:*

$$a\,(a + b) + a^2(a^2 + b^2) + a^3(a^3 + b^3) + \ldots\ldots\text{” provided } a < 1 \text{ and } b < 1.$$

Solution. We have $S_\infty = a\,(a + b) + a^2\,(a2 + b^2) + a^3\,(a^3 + b^3) + \ldots\ldots \infty$

$$= (a^2 + ab) + (a^4 + a^2b^2) + (a^6 + a^3b^3) + \ldots\ldots \infty$$

$$= (a^2 + a^4 + a^6 + \ldots\text{”}) + (ab + a^2b^2 + a^3b^3 + \ldots\ldots \infty$$

$$= \frac{a^2}{1-a^2} + \frac{ab}{1-ab}, \textit{sum of } \inf \operatorname{int} e\, G.P.'s$$

Geometric Mean

If a, G, b are in G.P., then G is the geometric mean (G.M.) between a and b or equivalently, if three quantities are in G.P. , then the middle one is called the G.M. between the other two.

$$\mathbf{G = \sqrt{ab}} \qquad \frac{\mathbf{G}}{a} = \frac{\mathbf{b}}{G} \ \textbf{i.e.,}\quad G^2 = ab$$

In general, if a, $G_1, G_2, G_3, \ldots\ldots, G_n$, b be a sequence of numbers which is a G.P., then $G_1, G_2, G_3, \ldots\ldots, G_n$ are the *n* geometric means between a and b.

To insert *n* G.M.'s between 'a' and 'b'

Suppose that $G_1, G_2, G_3, \ldots, G_n$ be the *n* G.M. between the two given numbers a and b.

So, a, $G_1, G_2, G_3, \ldots, G_n$, b are in G.P.

Let r be the common ratio of the G.P. Now there are n + 2 terms in this G.P. and b is the (n + 2)th term.

$T_{n+2} = a\, r^{n+1} = b \rightarrow (b/a)^{1/n+1}$

$G_1 = a\, r = a.\ (b/a)^{1/(n+1)}$ $G_2 = a\, r^2 = a\ .\ (b/a)^{2/(n+1)}$ …… $G_n = a\, r^n = a\ .\ (b/a)^{n/(n+1)}$ Which are the *n* G.M. between a and b.

Product of *n* G.M. between 'a' and 'b'

Here we are required to find the product of $G_1, G_2, G_3, \ldots, G_n$.

Now, $G_1 \times G_2 \times G_3 \times \ldots \times G_n$.

$= a\ .\ (b/a)^{1/(n+1)}\ .\ a\ .\ (b/a)^{2/(n+1)}\ .\ a\ .\ (b/a)^{3/(n+1)} \ldots\ldots\ a\ .\ (b/a)^{n/(n+1)}$

$= a^n.\ (b/a)^{(1+2+3+\ldots+n)/(n+1)} = a^n.\ (b/a)^{n(n+1)/2(n+1)} = a^n.(b/a)^{n/2} = (a^{1/2}.\ b^{1/2})^n$

$= (\sqrt{\ }\ ab)^n$ = *nth* power of the G.M. between a and b.

SOLVED EXAMPLES

Example1. *Insert 3 G.M.'s between*

(i) 3 and 432 *(ii) 1/9 and 9.*

Solution. (i) Let G_1, G_2, G_3 be the three G.M.'s between 3 and 432.

So, 3 , G_1, G_2, G_3, 432 are in G.P. Let r be the common ratio.

$T_5 = 3\, r^4 = 432\ \sqrt{\ }\ r^4 = 144 = (2\ \ 3)^4 \sqrt{\ }\ r = 2\ \ 3$

Now, $G_1 = a\, r = 3.\ 2\sqrt{\ }3 = 6\sqrt{\ }3$ $G_2 = a\, r^2 = 3.12 = 36$

$G_3 = a\, r^3 = 3.24\ \ 3 = 72\ \ 3$ Hence the required G.M.'s are **$6\sqrt{\ }3$, 36, $72\sqrt{\ }3$.**

(ii) Here, 1/9 , G_1, G_2, G_3, 9 are in G.P.

$T_5 = (1/9)\, r^4 = 9\ \sqrt{\ }\ r^4 = 81 = (3)^4 \sqrt{\ }\ r = 3$

Hence the required G.M.'s are ar, ar^2, ar^3

$G_1 = a\, r = (1/9) \times 3 = 1/3$ $G_2 = a\, r^2 = (1/9) \times 9 = 1$ $G_3 = a\, r^3 = (1/9) \times 27 = 3$

So, the required G.M.'s are **1/3, 1, 3.**

Example2. *The A.M. between two numbers is 20 and their G.M. is 16. Find the numbers.*

Solution. Suppose the two numbers a and b.

So, A.M. $= (a + b)/2 = 20$ '! $a + b = 40$ ….(1)

And, G.M. $= \sqrt{\ }\ ab = 16\ \sqrt{\ }\ ab = 256$ ….(2)

Now using the identity $(a - b)^2 = (a + b)^2 - 4ab$, we have

$(a-b)^2 = (40)^2 - 256$

→ $(a-b)^2 = 1600 - 1024 = 576$

→ $a - b = \pm 24.$ …(3)

Solving (1) and (3) , we get a = 32, b = 8 or a = 8, b = 32.

Hence, the required numbers are **8** and **32.**

1.14 Applications of Arithmetic and Geometric Progressions

Example 1: *For three consecutive months, a person deposits some amount of money on the first day of each month in small savings fund. These successive amounts in the fund deposited, the total value of which is Rs. 65, form a G.P.. If two extreme amounts be multiplied each by 3 and the mean by 5, the product forms an A.P. Find the amounts in the first and second deposits.*

Solution: Let a, ar, ar^2 be successive amounts deposited. Then

$a + ar + a r^2 = 65$ …….(i)

and 3a, 5ar and $3ar^2$ form an A.P. Thus, by the property of A.P., we have

$3a - 5ar = 5ar - 3ar^2$

$3ar^2 - 10ar + 3a = 0$

$3r^2 - 10r + 3 = 0$

$(r-3)(3r-1) = 0$, i.e., $r = 3$ or $1/3$.

For r = 3, from (i), we have; a + 3a + 9a = 65, i.e. 13a = 65 and a = 5. Thus, the amounts are Rs. 5, Rs. 15, and Rs. 45.

Again if r = 1/3 then from (i), we have [a + (a/3) + (a/9)] = 65, i.e., a = 45. Thus, the amounts are Rs.45, Rs. 15, and Rs. 5.

Hence, the amounts in the first and second deposits are Rs. 5, Rs. 15, and Rs. 45.

Example 2: *The value of a machine depreciated each year by 20%. If the percent value of the machine is Rs. 18,750, then find the estimated value at the end of 5 years.*

Solution: (i) Depreciation for the first year = 18,750 × 0.20. Thus, the depreciation value of the machine at the end of first year is : (18,750 – 18, 750 × 0.20) = 18,750(1– 0.20) = 15,000

(ii) Depreciation for the second year

= (Depreciation value at the end of first year) × (Rate of depreciation for second year)

= 18,750 × (1– 0.20) (0.20)

Depreciation value at the end of the second year is:

= (Depreciated value after first year) – (Depreciation for second year)

$= 18{,}750 \times (1-0.20) - 18{,}750 \times (1-0.20)(0.20) = 18{,}750 \times (1-0.20)^2$

Similarly, depreciated value of the machine at the end of third, fourth and fifth year is:

$18{,}750 \times (1-0.20)^3$ $18{,}750 \times (1-0.20)^4$ and $18{,}750 \times (1-0.20)^5$ respectively.

1.15. Summary

Quantitative techniques used to solve the complex problems that come in business or industrial area. For doing this we have many techniques. We are having many techniques according to the problem. For example to solve the problem related to profit and loss we are having profit function. For solve the problem of demand and supply we have demand and supply function, this function decides the cost of the product according to the demand of market, for finding revenue cost we have revenue cost function. For find out production from a raw material we have production function. The production function helps in decision making too. It can tell us that how many people we need to run an organization. How much capital we needed etc. By using consumption function we can estimate our consumption. Quantitative techniques not only help in business or economics. There are some special functions also. For example we have polynomial function, even and odd function, rational and irrational function, one to one function, onto function, into function, many to one, many to many, one one onto, one one into, and then we have composition of functions. This helps us in information technology field, as well as in banks and many other areas too.

It has a very important role in today's global economy. People are doing research in this area. It is the fastest growing technique for business sector. In every area functions helps us to solve our real problems. It also helps in proper deployment of resources, minimizing waiting and servicing costs, choosing an optimum strategy, optimum resources allocation, helps in decision making of management like in purchasing and decides the price of a particular product.

1.16 Key Words

- **Quantitative Technique:** It is a tool to help in decision making and problem solving.
- **Demand:** Quantity demanded of a particular commodity by the buyers.
- **Supply:** Handed particular commodity to the consumer that has been produced for them.
- **Cost:** Fixed and variable cost at the time of producing a particular commodity.
- **Revenue Function:** The total revenue in a particular commodity of a firm is depending on the number of units sold.
- **Profit:** Total Revenue – Total Cost.
- **A.P:** A sequence whose terms increase or decrease by a constant number called common difference.
- **G.P:** A sequence whose terms increase or decrease by a constant ratio called common ratio.

1.17 Self Assessment Test

1. Describe, in brief, some of the important quantitative techniques used in modern business industrial units.
2. Write a short note on "Operation Research" describing some of the important O.R. techniques.
3. Discuss fully the limitations of quantitative techniques.
4. Find the domain and range of the following functions:

   (i) $y = 1/(x-3)$ (ii) $y = 1/3x$
5. If $f(x) = 2x^2 + x + 1$, then simplify $f(x+1) - 2f(x) + f(x-1)$
6. Consider the quadratic equation $2x^2 - 8x + c = 0$. For what value of c the equation has

   (i) real roots (ii) equal roots and (iii) imaginary roots ?

7. Determine the quadratic equation whose roots are:

(i) – 1, –2 (ii) a/b, b/a (iii) – 3, –2/3

8. Find the point of equilibrium for the following demand – supply equations: Q = 5 – 3P and Q = 4P +12.

9. If $f(x) = \log x$, then proved that $f(x, y) = f(x) = f(y)$ and $f(x^n) = n f(x)$.

10. Graph the function $y = -x^2 + 4x - 2$ with the set of values $-5 \leq x \leq 5$ as the domain.

11. Find the quadratic function, $y = a x^2 + bx + c$, that fits the data points (1, 4), (–1, –2) and (2, 13). Estimate the value of y when x = 3.

12. The total cost curve for consumption of electricity is a linear function of the number of units consumed. The cost per unit on the first 60 units is slower and on the balance over 60, is higher. If the total cost for 100 units and 150 units consumed are Rs. 62 and Rs. 102, respectively, find the equation of the total cost line for consumption not less than 60 units and the charge for 180 units.

13. A manufacturer produces TV sets at a cost of Rs. 2,20,000 and 125 TV sets at a cost of 2,87,500. Assuming the cost curve to be linear, find the equation of this line and then use it to estimate the cost of 95 TV sets.

14. A firm produces an item that sells for Rs. 1.75each. IT costs the firm Rs. 0.75 to produce one unit of the item. The fixed costs are calculated to be Rs. 3000 per month. How many units would have to be produced each year to break-even? If the total number of capacity of the plant is 72,000 units, what is the break-even in the terms of percent capacity and rupees?

15. A television manufacturer finds that the total cost of producing television sets follows a quadratic function given by : $C(n) = 250n^2 + 3250n + 10,000$, where n denotes the number of television sets produced. Each product can be marketed for Rs. 6,500. Determine the break-even point.

16. Find the sum of the following series of the A.P.

(i) 2, 7 , 12, 17 …..

(ii) –5 –2 + 1 + 4 + ….to 20 terms.

(iii) 1, 2, 3, …

17. Show that the sum of (m + n)th term and (m–n)th term of A.P. is equal to the twice of mth term.

18. Find the increasing arithmetic progression, the sum of whose first three terms is 27 and sum of their square is 275.

19. In an A.P. which has odd number of terms. Prove that the middle term is the arithmetic mean of the first and last term.

20. The monthly salary of a person was Rs. 320 for each of the first three years. He next got annual increment of Rs. 40 per month for each of the following successive 12 years. His salary remained stationary till retirement when he found that his average monthly salary during the service period was Rs. 698. Find the period of his service.

21. Find (i) 6^{th} term of the series 5, 15, 45, …(ii) the nth term of the series 0.004, 0.02, 0.1, …

22. (i) If the first term of G.P. exceeds in 2^{nd} term by 2, and the sum to infinity is 50, find the series.

(ii) The product of three numbers in G.P. is 512. IF 8 is added to the first term and 6 is added to second, the number becomes in A.P. Find the numbers.

23. If S_n denotes the sum of the n terms of G.P. where the first term is 'a' and the common ratio is 'r'. Find the value of the sum $S_1 + S_3 + S_5 + S_7 + \ldots\ldots S_{2n-1}$.

24. A manufacturer records that the value of the machine costing him Rs. 8,800 depreciates at the rate of 25% per annum. Find the estimated value at the end of 5 years.

25. Annual increase in the population of a country is estimated to be 2%. What is the expected population at the end of the year 1988 if it is 75 crores at the end of the year 1978?

1.18 References

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Unit - 2 : Elementary Calculus and Matrix Algebra

Unit Structure:

2.0 Objectives

After reading this unit, you will be able to understand

- The meaning and types of functions
- Limits and continuity
- Application of limits to business problems (Maxima and Minima)
- Meaning and importance of matrices
- Types of matrices
- Matrix operations
- Determinants and their properties
- Gauss Jordan elimination method
- Application of matrices to business problems

2.1 Introduction

Calculus was developed independently by the Englishman, Sir Isaac Newton, and by the German, Gottfried Leibniz. From Leibniz we get the *dy/dx* and $\int$ signs.

Calculus is concerned with comparing quantities which vary in a *non-linear* way. It is used extensively in science and engineering since many of the things we are studying (like velocity, acceleration, and current in a circuit) do not behave in a simple, linear fashion.

Calculus is the study of change, in the same way that geometry is the study of shape and algebra is the study of operations and their application to solving equations.Calculus has a strong geometric flavor. Geometric space is nothing more than a collection of points, and geometric objects in space consist of some sub-collection of points in space.

2.2 Functions

A function is a technical term used to symbolise relationship between variables .When two variables are so related, that for any arbitrarily assigned value to one of them, there corresponds a definite value for the other.The area of a circle depends upon the length of its radius and therefore we could say that area is said to be the function of radius. The idea of function is expressed as –

The relationship between two real variables say x and y which are so related that corresponding to every value of x defined as the domain , we get a finite value b of y defined as the range then y is said to be the function of x .

The domain of variation of x in a function is called the domain of the function Function assigns exactly one output to each input. The argument and the value may be real numbers, but they can also be elements from any given set. A function f with argument x is denoted $f(x)$, which is read "f of x".

The number y is called the value of the function f at x and is written as $y = f(x)$

An example of such a function is $f(x) = 2x$, the function which associates with every number x the number twice as large. For instance, if its argument is 5 its value is 10, and this is written as:

$f(5) = 10.$

• **<u>Types of functions:</u>**

a) Explicit and implicit functions:

The **explicit function** is a function expressed directly in terms of the dependent variable is said to be an explicit function .It is denoted by: y= f(x)

Example :

$Y = ax^n+bx$ where a, n and b are constant. $y= 5x^3-3$

The **Implicit function** is a function which is not expressed directly in terms of of the dependent variable, there is a mutual relationship between two variables and either variable determines the other, is said to be implicit function .It is denoted by: R(x,y) = 0

b) **Single valued and many valued functions:**

A **single-valued function** is each element of the function's domain maps to a single, well-defined element of its range. This contrasts with a general binary relation, which can be viewed as being a multi-valued function.

For example: $f(x) = x+3$ (each element in domain has not more than one image in range set).

A **multivalued function** is a left-total relation, i.e. every input is associated with one or more outputs.. Strictly speaking, a "well-defined" function associates one, and only one, output to any particular input.

c) Even and odd functions: Function of a real variable. A function f is **even** if the following equation holds for all x in the domain of f: $f(x)=f(-x)$

Examples of even functions are $|x|$, x^2, x^4, cos(x), and cos(x)

Again, let $f(x)$ be a real valued function of a real variable. A function f is **odd** if the following equation holds for all x in the domain of f: $-f(x) = f(-x)$, or $f(x)+f(-x)=0$

d) **Parametric functions:** If the value of x and y can be expressed in terms of another variable t, as x= f(t),y=(t).

e) **Continous and Discontinous functions** : A function is said to be continous in a given interval (a,b) provided if we represent the function graphically for different values, and the function does not break, it is said to be continous one , otherwise discontinous.

- **Examples of Functions in Economics**

Demand Function: If p is the price and x is the quantity of a commodity demanded by the consumers

$p = f(x)$ *or* $x = f(p)$ where q is the quantity demanded and p is the price.

Supply Function: If p is the price and x is the quantity of a commodity supplied by the firm , we write the supply function as $q = f(p)$ *or* $x = f(p)$ where q is the quantity of a commodity supplied by a firm.

Production function: If x is the amount of output produced by the firm, when the factors or inputs land (D), labour (l), capital (k) and organization (o)are employed . We write the production function as

$q = f(D, l, k, o)$ where q is the amount produced by the firm.

Utility function: If U is utility of a consumer and $q1$ and $q2$, are the quantities of commodities consumed, we write the utility function as $U = f(q1, q2)$

Cost Function: If x is the quantity produced by a firm at a total cost C, we write for cost function as $C= f(x)$

2.3 Limits and Continuity

- **Limits**

Definition. The limit of a function is that fixed value to which a function approaches as the variable approaches a given value. The concept of limit is very important concept in calculus. A particular case of a limit leads to continuity. To says that:

$$\lim_{x \to p} f(x) = L.$$

Means that $f(x)$ can be made as close as desired to L by making x close enough, but not equal, to p.

- **Continuity**

A function is continuous if you can draw its graph without taking your pencil off the page. But sometimes this will be true for some parts of a graph but not for others. Therefore, we want to start by defining what it means for a function to be continuous at *one point*.

Definition: (continuity at a point)

If $f(x)$ is defined on an open interval containing c, then $f(x)$ is said to be **continuous at c** if and only if: $\lim_{x \to c} f(x) = f(c)$

Note that for f to be continuous at c, the definition in effect requires three conditions:

1. that f is defined at c, so $f(c)$ exists,
2. the limit as x approaches c exists, and
3. The limit and $f(c)$ are equal.

If any of these do not hold then f is not continuous at c.

2.4 Differentiation

In calculus a branch of mathematics the **derivative** is a measure of how a function changes as its input changes. In other words a derivative can be thought of as how much one quantity is changing in response to changes in some other quantity; for example, the derivative of the position of a moving object with respect to time is the object's instantaneous velocity.

The process of finding a derivative is called **differentiation**. The reverse process is called **anti-differentiation**. The fundamental theorem of calculus states that anti-differentiation is the same as integration.

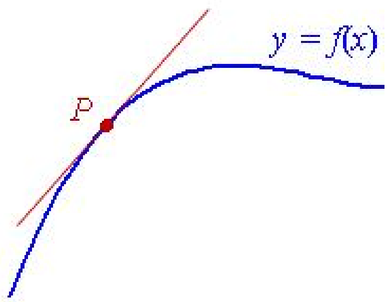
• The slope of a tangent to a curve

Applications include:

- Temperature change at a particular time
- Velocity of a falling object at a particular time
- Current through a circuit at a particular time
- Variation in stock market prices at a particular time
- Population growth at a particular time
- Temperature increase as density increases in a gas

Now, we will find rates of change **numerically** (that is, by substituting numbers in until we find an acceptable approximation.)

We look at the general case and write our functions involving the familiar x (independent) and y (dependent) variables.



The slope of a **curve** $y = f(x)$ at the point P means the slope of the **tangent** at the point P. We have to find this slope to solve many applications since it tells us *the rate of change* at a particular instant.

[We write $y = f(x)$ on the curve since y is a function of x. That is, as x varies, y varies also.]

Delta Notation

In this , we write

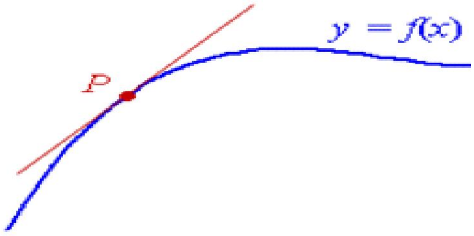
- **change in *y*** as **Δy**
- **change in *x*** as **Δ*x***

By definition, the slope is given by:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

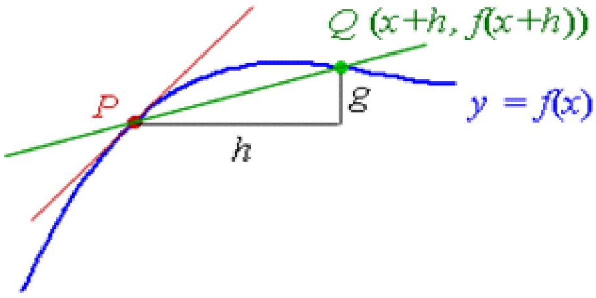
• The derivative from the first principle (Delta Method)

First principles is also known as "delta method", since many texts use Δx (for "change in x) and Δy (for "change in y"). This makes the algebra appear more difficult, so here we use h for Δx instead. We still call it "delta method".



We wish to find an **algebraic method** to find the slope of y = f(x) at *P*, to save doing the numerical substitutions that we saw in the last section (Slope of a Tangent to a Curve).

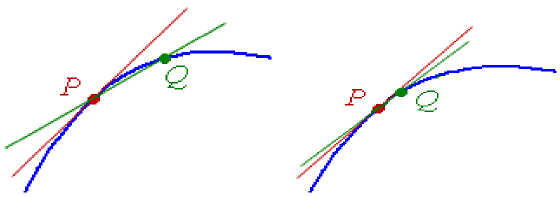
We can approximate this value by taking a point somewhere near to P(x, f(x)), say Q(x + h, f(x + h)).



The value g/h is an approximation to the slope of the tangent which we require.

We can also write this slope as "change in *y* / change in *x*" or: $m = \frac{\Delta y}{\Delta x}$

If we move *Q* closer and closer to *P*, the line *PQ* will get closer and closer to the tangent at *P* and so the slope of *PQ* gets closer to the slope that we want.



If we let *Q* go all the way to touch *P* (i.e. *h* = 0), then we would have the **exact** slope of the tangent.

Now, g/h can be written: $\frac{g}{h} = \frac{f(x+h) - f(x)}{h}$

So also, the slope *PQ* will be given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

But we require the slope **at** *P*, so we let $h \to 0$ (that is let *h* approach 0), then in effect, *Q* will approach *P* and g/h will approach the required slope.

Putting this together, we can write the slope of the tangent at *P* as:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is called **differentiation from first principles,** (or the **delta method**). It gives the instantaneous rate of change of y with respect to x.

This is equivalent to the following (where before we were using h for Δx):

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

You will also come across the following for delta method:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Notation for the Derivative

Important: The *derivative* (also called **differentiation**) can be written in several ways. This can cause some confusion when we first learn about differentiation.

The following are equivalent ways of writing the first derivative of $y = f(x)$:

$\frac{dy}{dx}$ or $f'(x)$ or y'.

Example :

Find $\frac{dy}{dx}$ from first principles if $y = 2x^2 + 3x.7$

Solution :

$f(x) = 2x^2 + 3x$ so

$$\begin{aligned} f(x+h) &= 2(x+h)^2 + 3(x+h) \\ &= 2(x^2 + 2xh + h^2) + 3x + 3h \\ &= 2x^2 + 4xh + 2h^2 + 3x + 3h \end{aligned}$$

We now need to find:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[2x^2 + 4xh + 2h^2 + 3x + 3h] - [2x^2 + 3x]}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 + 3h}{h} \\ &= \lim_{h \to 0} 4x + 2h + 3 \\ &= 4x + 3 \end{aligned}$$

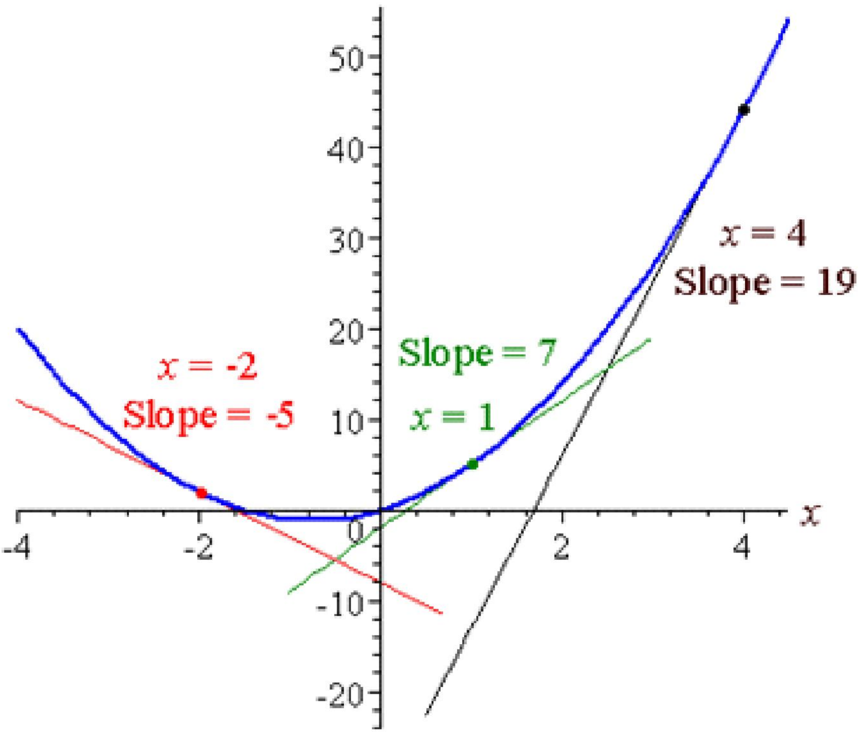
We have found an expression that can give us the slope of the tangent anywhere on the curve.

If $x = -2$, the slope is $4(-2) + 3 = -5$ (red, in the graph below)

If $x = 1$, the slope is $4(1) + 3 = 7$ (green)

If $x = 4$, the slope is $4(4) + 3 = 19$ (black)

We can see that our answers are correct when we graph the curve (which is a parabola) and observe the slopes of the tangents.



This is what makes calculus so powerful. We can find the slope anywhere on the curve (i.e. the rate of change of the function anywhere).

• Definition Via Difference Quotients

Let f be a real valued function. In classical geometry, the tangent line to the graph of the function f at a real number a was the unique line through the point $(a, f(a))$ that did *not* meet the graph of f transversally, meaning that the line did not pass straight through the graph. The derivative of y with respect to x at a is, geometrically, the slope of the tangent line to the graph of f at a. The slope of the tangent line is very close to the slope of the line through $(a, f(a))$ and a nearby point on the graph, for example $(a + h, f(a + h))$. These lines are called secant lines. A value of h close to zero gives a good approximation to the slope of the tangent line, and smaller values (in absolute value) of h will, in general, give better approximations. The slope m of the secant line is the difference between the y values of these points divided by the difference between the x values,

that is, $$m = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}.$$

This expression is Newton's difference quotient. The derivative is the value of the difference quotient as the secant lines approach the tangent line. Formally, the **derivative** of the function f at

a is the limit: $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

of the difference quotient as h approaches zero, if this limit exists. If the limit exists, then f is **differentiable** at a. Here $f2\ (a)$ is one of several common notations for the derivative (see below).

Equivalently, the derivative satisfies the property that: $$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a) \cdot h}{h} = 0,$$

Which has the intuitive interpretation that the tangent lines to f at a gives the *best linear* *approximation:* $f(a+h) \approx f(a) + f'(a)h$

f near *a* (i.e., for small *h*). This interpretation is the easiest to generalize to other settings (see below).

Substituting 0 for *h* in the difference quotient causes division by zero, so the slope of the tangent line cannot be found directly using this method. Instead, define $Q(h)$ to be the difference quotient

as a function of *h*: $Q(h) = \frac{f(a+h) - f(a)}{h}$.

$Q(h)$ is the slope of the secant line between $(a, f(a))$ and $(a + h, f(a + h))$. If *f* is a continuous function, meaning that its graph is an unbroken curve with no gaps, then *Q* is a continuous function away from $h = 0$. If the limit exists, meaning that there is a way of choosing a value for $Q(0)$ that makes the graph of *Q* a continuous function, then the function *f* is differentiable at *a*, and its derivative at *a* equals $Q(0)$.

- **Example**

The squaring function $f(x) = x^2$ is differentiable at $x = 3$, and its derivative there is 6. This result is established by calculating the limit as *h* approaches zero of the difference quotient of $f(3)$:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h\to 0} \frac{6h + h^2}{h} = \lim_{h\to 0} (6 + h).$$

The last expression shows that the difference quotient equals $6 + h$ when $h \neq 0$ and is undefined when $h = 0$, because of the definition of the difference quotient. However, the definition of the limit says the difference quotient does not need to be defined when $h = 0$. The limit is the result of letting *h* go to zero, meaning it is the value that $6 + h$ tends to as *h* becomes very small:

$$\lim_{h\to 0} (6 + h) = 6 + 0 = 6.$$

Hence the slope of the graph of the squaring function at the point (3, 9) is 6, and so its derivative at $x = 3$ is f '$(3) = 6$.

More generally, a similar computation shows that the derivative of the squaring function at $x = a$ is f '$(a) = 2a$.

- **The Derivative as a Function**

Let *f* be a function that has a derivative at every point *a* in the domain of *f*. Because every point *a* has a derivative, there is a function that sends the point *a* to the derivative of *f* at *a*. This function is written *f2 (x)* and is called the *derivative function* or the *derivative* of *f*. The derivative of *f* collects all the derivatives of *f* at all the points in the domain of *f*.

Sometimes *f* has a derivative at most, but not all, points of its domain. The function whose value at *a* equals *f2 (a)* whenever *f2 (a)* is defined and elsewhere is undefined is also called the derivative of *f*. It is still a function, but its domain is strictly smaller than the domain of *f*.

The derivative is an operator whose domain is the set of all functions that have derivatives at every point of their domain and whose range is a set of functions. If we denote this operator by *D*, then $D(f)$ is the function *f2* (x). Since $D(f)$ is a function, it can be evaluated at a point *a*. By the definition of the derivative function, $D(f)(a) =$ *f2* (a).

For comparison, consider the doubling function $f(x) = 2x$; *f* is a real-valued function of a real number, meaning that it takes numbers as inputs and has numbers as outputs:

$$\begin{aligned} 1 &\mapsto 2, \\ 2 &\mapsto 4, \\ 3 &\mapsto 6. \end{aligned}$$

The operator D, however, is not defined on individual numbers. It is only defined on functions:

$$\begin{aligned} D(x \mapsto 1) &= (x \mapsto 0), \\ D(x \mapsto x) &= (x \mapsto 1), \\ D(x \mapsto x^2) &= (x \mapsto 2 \cdot x). \end{aligned}$$

Because the output of D is a function, the output of D can be evaluated at a point. For instance, when D is applied to the squaring function

$$x \mapsto x^2,$$

D outputs the doubling function,

$$x \mapsto 2x,$$

which we named $f(x)$. This output function can then be evaluated to get $f(1) = 2$, $f(2) = 4$, and so on.

• Inflection Point

A point where the second derivative of a function changes sign is called an **inflection point**. At an inflection point, the second derivative may be zero, as in the case of the inflection point $x=0$ of the function $y=x^3$, or it may fail to exist, as in the case of the inflection point $x=0$ of the function $y=x^{1/3}$. At an inflection point, a function switches from being a convex function to being a concave function or vice versa.

• Computing the Derivative

The derivative of a function can, in principle, be computed from the definition by considering the difference quotient, and computing its limit. In practice, once the derivatives of a few simple functions are known, the derivatives of other functions are more easily computed using *rules* for obtaining derivatives of more complicated functions from simpler ones.

• Derivatives of Elementary Functions

Most derivative computations eventually require taking the derivative of some common functions. The following incomplete list gives some of the most frequently used functions of a single real variable and their derivatives.

- *Derivatives of powers*: if $f(x) = x^r$, where r is any real number, then

$$f'(x) = rx^{r-1},$$

wherever this function is defined. For example, if $f(x) = x^{1/4}$, then $f'(x) = (1/4)x^{-3/4}$,

Exponential and logarithmic functions:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = \ln(a)a^x$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \qquad x > 0$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

- **Rules for Finding the Derivative**

In many cases, complicated limit calculations by direct application of Newton's difference quotient can be avoided using differentiation rules. Some of the most basic rules are the following.

- $$\frac{dc}{dx} = 0$$

- *Sum rule*: $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Here, *u* and *v* are functions of *x*. The derivative of the sum is equal to the derivative of the first plus derivative of the second.

- *Product rule*:

If *u* and *v* are two functions of *x*, then the derivative of the product *uv* is given by...

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In words, this can be remembered as:

"The derivative of a product of two functions is the first times the derivative of the second, plus the second times the derivative of the first."

Example: If *u* and *v* are two functions of *x*, then the derivative of the product *uv* is given by...

If we have a product like

$y = (2x^2 + 6x)(2x^3 + 5x^2)$

we can find the derivative without multiplying out the expression on the right.

Solution: We use the substitutions $u = 2x^2 + 6x$ and $v = 2x^3 + 5x^2$.

We can then use the product rule:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

We first find: $\frac{dv}{dx} = 6x^2 + 10x$ and $\frac{du}{dx} = 4x + 6$

Then we can write:

$$\begin{aligned}\frac{d(uv)}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (2x^2 + 6x)(6x^2 + 10x) + (2x^3 + 5x^2)(4x + 6)\\ &= 20x^4 + 88x^3 + 90x^2\end{aligned}$$

- *Quotient rule*:

(A **quotient** is just a fraction.)

If *u* and *v* are two functions of *x*, then the derivative of the quotient *u/v* is given by...

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

In words, this can be remembered as:

"The derivative of a quotient equals bottom times derivative of top minus top times derivative of the bottom, divided by bottom squared."

Example: We wish to find the derivative of the expression: $y = \frac{2x^3}{4 - x}$

Solution: We recognize that it is in the form: y = u/v**.**

We can use the substitutions:

$u = 2x^3$ and $v = 4$ " x

Using the **quotient rule,** we first need to find:

$\frac{du}{dx} = 6x^2$ And $\frac{dv}{dx} = -1$ then

$$\begin{aligned}\frac{d(\frac{u}{v})}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(4-x)(6x^2) - (2x^3)(-1)}{(4-x)^2}\\ &= \frac{24x^2 - 6x^3 + 2x^3}{(4-x)^2}\\ &= \frac{24x^2 - 4x^3}{(4-x)^2}\end{aligned}$$

Chain rule: **It** is a formula for computing the derivative of the composition of two or more functions.

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example: Find (3x)^2

$$\begin{aligned}\frac{dy}{dx}(3x)^2 = 2(3x)^{(2-1)} * \frac{dy}{dx}(3x)\\ = 2(3x)(3)\\ = 6(3x)\\ = 18x\end{aligned}$$

2.5 Application of Calculus to Business

Let's take a look at some applications of derivatives in the business world. For the most part these are really applications that we've already looked at, but they are now going to be approached with an eye towards the business world.

Let's start things out with a couple of optimization problems. We've already looked at more than a few of these in previous sections so there really isn't anything all that new here except for the fact that they are coming out of the business world.

Example 1. An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by, $P(x) = -8x^2 + 3200x - 80,000$

How many apartments should they rent in order to maximize their profit?

Solution. All that we're really being asked to do here is to maximize the profit subject to the constraint that x must be in the range.

First, we'll need the derivative and the critical point(s) that fall in the range. $0 \le x \le 250$

The derivative p'(x) of the the profit equation p(x) is as follows:

$$
\begin{aligned}
P'(x) &= -16x+3200 \\
&= 3200 - 16x = 0 \\
&= 3200/16 \\
&= 200
\end{aligned}
$$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have (which nicely enough in the range of acceptable answers) and the end points of the range.

Substituting values in the original function i.e..

$$
\begin{aligned}
P(0) &= -80,000 \\
&= 2,40,000 \\
P(250) &= 2,20,000
\end{aligned}
$$

So, it looks like they will generate the most profit if they only rent out 200 of the apartments instead of all 250 of them.

Example 2. A production facility is capable of producing 60,000 pens in a day and the total daily cost of producing pens x in a day is given by, $C(x) = 250,000 + 0.08x + \frac{200,000,000}{x}$

How many pens per day should they produce in order to minimize production costs?

Solution

Here we need to minimize the cost subject to the constraint that x must be in the range .$0 \le x \le 60,000$ Note that in this case the cost function is not continuous at the left endpoint and so we won't be able to just plug critical points and endpoints into the cost function to find the minimum value.

Let's get the first couple of derivatives of the cost function.

$$C'(x) = 0.08 - \frac{200,000,000}{x^2} \qquad\qquad C''(x) = \frac{400,000,000}{x^3}$$

The critical points of the cost function are,

$$
\begin{aligned}
0.08 - \frac{200,000,000}{x^2} &= 0 \\
0.08x^2 &= 200,000,000 \\
x^2 &= 2,500,000,000 \quad \Rightarrow \quad x = \pm\sqrt{2,500,000,000} = \pm 50,000
\end{aligned}
$$

Now, clearly the negative value doesn't make any sense in this setting and so we have a single critical point in the range of possible solutions: 50,000.

Now, as long as the second derivative is positive and so, in the range of possible solutions the function is always concave up and so producing 50,000 pens will yield the absolute minimum production cost.

Example 3. The production costs per day for some widget is given by,

$$C(x) = 2500 - 10x - 0.01x^2 + 0.0002x^3$$

What is the marginal cost when , and ?

Solution

So, we need the derivative and then we'll need to compute some values of the derivative.

$$C'(x) = -10 - 0.02x + 0.0006x^2$$

$$C'(200) = 10 \qquad C'(300) = 38 \qquad C'(400) = 78$$

So, in order to produce the 201st widget it will cost approximately $10. To produce the 301st widget will cost around $38. Finally, to product the 401st widget it will cost approximately $78.

2.6 An Introduction to Matrices

The knowledge of matrices is required in many areas of business and other spheres of life.In this section we shall study the fundamentals of matrix algebra.

Suppose we wish to express that Shyam has 10 books. We may express it as (10) with the understanding that the number inside ( ) denotes the number of books that Shyam has. Next suppose we want to express that Shyam has 10 books and 5 pens . We may express it as (10 5) with the understanding that the first entry inside ( ) denotes the number of books while the second entry, the number of pens, possessed by Shyam . Let us now suppose that we have three children Shyam , Krishna and Mohan . Suppose Shyam has 10 books, 5 pens, Krishna has 4 books, 6 pens and Mohan has 5 books, 8 pens. This information can be represented in the tabular form as follows:

	Books	Pens
Shyam	10	5
Krishna	4	6
Mohan	5	8

We can also briefly write these as follows:

	First Column	Second column
First Row	10	5
Second Row	4	6
Third row	5	8

The Information behind can be expresse3d in two rows and three columns as given below:

	Shyam	Krishna	Mohan
Books	10	4	5
Pens	5	6	8

Definition:

A matrix is an array of numbers. A matrix with m rows and n columns is order m x n and is shown as follows..

$$\begin{bmatrix} a_{11} & a_{12} \cdots & \cdots a_{1n} \\ a_{21} & a_{22} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot \quad a_{ij} & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} \cdots & \cdots a_{mn} \end{bmatrix} = [A]$$

2.7 Types of Matrices

In this section we are going to discuss about **types of matrices** concept. In mathematics, a **matrix** is a rectangular array of numbers, such as $\begin{bmatrix} 1 & 9 & 13 \\ 20 & 55 & 4 \end{bmatrix}$

An item in a matrix is called an entry or an element. The example has entries 1, 9, 13, 20, 55, and 4. Entries are often denoted by a variable with two subscripts, as shown on the right. Matrices of the same size can be added and subtracted entry wise and matrices of compatible sizes can be multiplied.

- A **row matrix** has one row of numbers as shown below:-

$$[a_1 \quad a_2 \, . \, . \, . \, a_n] = [A]$$

- A **column matrix** has one column of numbers as shown below:-

$$\begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_m \end{bmatrix} = \{A\}$$

- A **square matrix** is one with an equal number of rows and columns i.e m = n

For example,

$$\begin{pmatrix} 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 5 \\ 0 & 6 & 1 \\ 7 & 5 & 2 \end{pmatrix}$$

- A **diagonal matrix** is a square matrix with all numbers zero apart from diagonal numbers as shown below:-

$$\begin{bmatrix} a_{11} & 0 & 0 & . & 0 \\ 0 & a_{22} & 0 & . & . \\ 0 & . & . & . & . \\ . & . & . & . & a_{nn} \end{bmatrix}$$

- A **unit matrix (also called as Identity Matrix)** is a square matrix with all diagonal numbers = 1-. The other elements being 0..

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I]$$

- A **null matrix/Zero Matrix** has all elements = 0

For example,

$$\begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2.8 Matrix Operations

In this section, we shall define equality of matrices, multiplication of a matrix by a scalar , addition, subtraction and multiplication of matrices.

- **Equality of matrices**

Definition: Two matrices A= (a_{ij}) and B= (b_{ij}) are said to be equal if

i) they are of same order and
ii) each element of A is equal to the corresponding element of B i.e., $a_{ij} = b_{ij}$ for all i and j

For example, if
$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 1 & 0 \\ 3 & 4 \end{pmatrix}$$

Then a =5, b =-1, c =1, d =0, e = 3, f = 4

For Example

If
$$\begin{pmatrix} x-y & z \\ 2x-y & w \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & 5 \end{pmatrix}$$

Find x, y, z, w.

Solution. By definition of equality of matrices , we have

$x-y = -1$ (1)
$2x-y = 0$ (2)
$z = 4$ (3)
$w = 5$ (4)
Solving (1) and (2), we get

X = 1, y = 2

Hence x = 1, y = 2, z = 4, w = 5

- **Multiplication of a matrix by a scalar**

Definition: If A = (a_{ij}) is a matrix and k is a scalar , then kA is another matrix whose (i, j)th element is $k\,a_{ij}$ for all possible values of i and j.

Thus, $k\,A = (k\,a_{ij})$

Example:

If A = $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \\ 6 & 0 & 2 \end{pmatrix}$, find 3A.

Solution. We have

$$3A = 3\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \\ 6 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 9 \\ 12 & 15 & -3 \\ 18 & 0 & 6 \end{pmatrix}$$

• **Addition of Matrices**

Definition: The expression $z_{ij} = a_{ij} + b_{ij}$ means "to element in row i, column j of matrix **A** add element in row i, column j of matrix **B**". If we do this with each element of **A** and **B** we end with matrix **Z**. An example is given in

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 5 & 8 \\ -2 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -6 & 1 \\ 0 & 9 \\ 8 & 3 \end{bmatrix} \quad \mathbf{Z = A + B} = \begin{bmatrix} -3 & 7 \\ 5 & 17 \\ 6 & 12 \end{bmatrix}$$

$$z_{ij} = a_{ij} + b_{ij}$$

$$z_{11} = 3 + -6 = -3 \qquad z_{12} = 6 + 1 = 7$$

$$z_{21} = 5 + 0 = 5 \qquad z_{22} = 8 + 9 = 17$$

$$z_{31} = -2 + 8 = 6 \qquad z_{32} = 9 + 3 = 12$$

Figure

The above figure shows addition operation ,in the same manner the subtraction is performed in analogous manner. The expression $z_{ij} = a_{ij} - b_{ij}$, means "to element in row i, column j of matrix **A** deduct element in row i, column j of matrix **B**". If we do this with each element of **A** and **B** we end with matrix **Z**.

• **Subtraction of Matrices**

In the above example if we want to find the subtraction between the two matrices we will perform the following operation:

$$A = \begin{pmatrix} 3 & 6 \\ 6 & 5 \\ 8 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 4 & 4 \\ 5 & 1 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$$

- **Multiplication of Matrices**

Definition:Consider two matrices **A** and **B** with the following characteristics: the number of columns in **A** equals the number of rows in **B**. These are *conformable* with respect to one another, and they can be multiplied together to form a new matrix **Z**.

The expression zij = $a_{i1} * b_{1j} + a_{i2} * b_{2j} + a_{i3} * b_{3j} + \ldots a_{im} * b_{nj}$, means "add the products obtained by multiplying elements in each i row of matrix **A** by elements in each j column of matrix **B**". Figure 2.6 shows what we mean by this statement.

$$A = \begin{bmatrix} 4 & 1 & 9 \\ 6 & 2 & 8 \\ 7 & 3 & 5 \\ 11 & 10 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 9 \\ 5 & 12 \\ 8 & 10 \end{bmatrix} \quad Z = A^*B = \begin{bmatrix} 85 & 138 \\ 86 & 158 \\ 69 & 149 \\ 168 & 339 \end{bmatrix}$$

$$z_{ij} = a_{i1} * b_{1j} + a_{i2} * b_{2j} + a_{i3} * b_{3j} + \ldots a_{im} * b_{nj}$$

$z_{11} = 4*2 + 1*5 + 9*8 = 85$ $\qquad z_{12} = 4*9 + 1*12 + 9*10 = 138$

$z_{21} = 6*2 + 2*5 + 8*8 = 86$ $\qquad z_{22} = 6*9 + 2*12 + 8*10 = 158$

$z_{31} = 7*2 + 3*5 + 5*8 = 69$ $\qquad z_{32} = 7*9 + 3*12 + 5*10 = 149$

$z_{41} = 11*2 + 10*5 + 12*8 = 168$ $\qquad z_{42} = 11*9 + 10*12 + 12*10 = 339$

Figure

The order in which we multiply terms does matter. The reason for this is that we need to multiply row elements by column elements and one by one. Therefore **A*B** and **B*A** can produce different results. We say "can produce" because there exist special cases in which the operation is commutative (order does not matter). .

2.9 Determinants

The *determinant* of a *square* matrix **A** = [*ai j*] is a number denoted by |**A**| or det*(***A***)*, through which important properties such as singularity can be briefly characterized. This number is defined as the following function of the matrix elements:

$$|\mathbf{A}| = \det(\mathbf{A}) = a1\, j1\; a2\, j2 \ldots an\, jn,$$

where the column indices *j*1, *j*2, . . . *jn* are taken from the set {1, 2, . . . *n*}, with no repetitions allowed. The plus (minus) sign is taken if the permutation *(j*1 *j*2 . . . *jn)* is even (odd).

To indicate that we are referring to determinant **A** and not to matrix **A** we surround the symbol A by pipes

(“|”).

Determinants are like matrices, but done up in absolute-value bars instead of square brackets. There is a lot that you can do with (and learn from) determinants, but you’ll need to wait for an advanced course to learn about them.

• **Determinant Order.**

Determinant of a 2 by 2 Matrix

The determinant of the 2 by 2 matrix A

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a scalar given by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a*d - c*b$$

Example : Find the determinant of the 2 by 2 matrix A given by

$$A = \begin{bmatrix} -2 & 3 \\ 5 & 1/2 \end{bmatrix}$$

Solution:

det(A) = -2*(1/2)-5*3
=-16

• Determinant of a 3 by 3 Matrix

The determinant of the 3 by 3 matrix A

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is a scalar given by

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a(ei - hf) - d(bi - hc) + g(bf - ec)$$

Example : Find the determinant of the 3 by 3 matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 5 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

Solution

det (A) = 2(5*5-4*1) - (-1)*(-2*5 - 4*0) + 3*(-2*1 - 5*0) = 42 - 10 -6 = 26

• Expansion using Minors and Cofactors

The definition of determinant that we have so far is only for a 2×2 matrix. There is a shortcut for a 3×3 matrix, but I firmly believe you should learn the way that will work for all sizes, not just a special case for a 3×3 matrix.

The method is called expansion using minors and cofactors. Before we can use them, we need to define them.

Minors

A minor for any element is the determinant that results when the row and column that element are in are deleted.

The notation M_{ij} is used to stand for the minor of the element in row i and column j. So M_{21} would mean the minor for the element in row 2, column 1.

Consider the 3×3 determinant shown below. I've included headers so that you can keep the rows and columns straight, but you wouldn't normally include those. We're going to find some of the minors

C , C_2 , C_3.

Finding the Minor for R_2C_1

The minor is the determinant that remains when you delete the row and column of the element you're trying to find the minor for. That means we should delete row 2 and column 1 and then find the determinant.

$$\begin{array}{cc} & \begin{matrix} C_1 & C_3 \end{matrix} \\ \begin{matrix} R_1 \\ R_3 \end{matrix} & \begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} \end{array} = 3(2)-5(2) = 6-10 = -4$$

As you can see, the minor for row 2 and column 1 is $M_{21} = -4$.

Let's try another one.

Finding the Minor for R_3C_2

This time, we would delete row 3 and column 2.

$$\begin{array}{cc} & \begin{matrix} C_1 & C_3 \end{matrix} \\ \begin{matrix} R_1 \\ R_2 \end{matrix} & \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} \end{array} = 1(3) - 4(2) = 3 - 8 = -5$$

So the minor for row 3, column 2 is $M_{32} = -5$.

• ***Co-Factor of Determinant element***

Associated with each element is a sign (+ or -) this is established by counting the number of row and column steps to get from the first element and if the steps are 0 or even the sign is (+) and if the steps are odd the sign is (-). e.g.

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

The co-factor of and element is the minor of the element x the relevant sign. e.g.

The co-factor of $a_1 = A_1 = + \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$ The co-factor of $b_3 = B_3 = - \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_4 & c_4 & d_4 \end{vmatrix}$

• ***Expansion of Determinant***

The expansion of a determinant is completed by adding the product of all elements with their co-factors for any row or column. eg

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1.A_1 - b_1.B_1 + c_1.C_1$$

An example of the expansion of a determinant is provided below..

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & -3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= (1+12) - 2\,(3-8) + 3\,(-9+2) = 13 + 10 - 33 = -10$$

2.10 Properties of Determinants

In this section, we will study properties determinants have and we will see how these properties can help in computing the determinant of a matrix. We will also see how these properties can give us information about matrices.

1. $|A^t| = |A|$

The determinant of matrix A and its transpose A^t are equal.

$$A = \begin{vmatrix} 2 & 3 & 0 \\ 3 & 2 & 7 \\ 2 & 1 & 6 \end{vmatrix} \quad A^t = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 0 & 7 & 6 \end{vmatrix}$$

$$|A| = |A^*| = -2$$

2. **|A|= 0 If:**

(i) It has two equal lines

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

(ii) All elements of a line are zero.

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(iii) The elements of a line are a linear combination of the others.

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$$

$\mathbf{r_3 = r_1 + r_2}$

3.A triangular determinant is the product of the diagonal elements.

$$A=\begin{vmatrix}2&0&0\\1&2&0\\3&5&6\end{vmatrix}=2\cdot2\cdot6=24$$

4. If a determinant switches two parallel lines its determinant changes sign.

$$\begin{vmatrix}2&1&2\\1&2&0\\3&5&6\end{vmatrix}=-\begin{vmatrix}1&2&0\\2&1&2\\3&5&6\end{vmatrix}$$

5. If the elements of a line are added to the elements of another parallel line previously multiplied by a real number, the value of the determinant is unchanged.

$$\begin{vmatrix}2&1&2\\1&2&0\\3&5&6\end{vmatrix}=16 \qquad C_3=2C_1+C_2+C_3 \qquad \begin{vmatrix}2&1&7\\1&2&4\\3&5&17\end{vmatrix}=16$$

6. If a determinant is multiplied by a real number, any line can be multiplied by the above mentioned number, but only one.

$$2\cdot\begin{vmatrix}2&1&2\\1&2&0\\3&5&6\end{vmatrix}=\begin{vmatrix}2\cdot2&1&2\\1\cdot2&2&0\\3\cdot2&5&6\end{vmatrix}=\begin{vmatrix}4&1&2\\2&2&0\\6&5&6\end{vmatrix}$$

7. If all the elements of a line or column are formed by two addends, the above mentioned determinant decomposes in the sum of two determinants.

$$\begin{vmatrix}2&1&2\\a+b&a+c&a+d\\3&5&6\end{vmatrix}=\begin{vmatrix}2&1&2\\a&a&a\\3&5&6\end{vmatrix}+\begin{vmatrix}2&1&2\\b&c&d\\3&5&6\end{vmatrix}$$

8. **|A·B| =|A|·|B|**

The determinant of a product equals the product of the determinants.

Examples

1. Apply the properties of determinants and calculate:

$$A=\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} \qquad B=\begin{vmatrix}1&0&1\\0&1&0\\0&0&1\end{vmatrix} \qquad C=\begin{vmatrix}2&3&4\\2&a+3&b+4\\2&c+3&d+4\end{vmatrix}$$

$$A=\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}=\begin{matrix}\xrightarrow{r_2-r_1}\\ \\ \xrightarrow{r_3-r_2}\end{matrix}\begin{vmatrix}1&2&3\\3&3&3\\3&3&3\end{vmatrix}=0$$

$$B=\begin{vmatrix}1&0&1\\0&1&0\\0&0&1\end{vmatrix}=1\cdot\begin{vmatrix}1&0\\0&1\end{vmatrix}=1\cdot1=1$$

$$C=\begin{vmatrix}2&3&4\\2&a+3&b+4\\2&c+3&d+4\end{vmatrix}=\begin{matrix}\xrightarrow{r_2-r_1}\\ \\ \xrightarrow{r_3-r_1}\end{matrix}\begin{vmatrix}2&3&4\\0&a&b\\0&c&d\end{vmatrix}=2\begin{vmatrix}a&b\\c&d\end{vmatrix}=2(ad-bc)$$

2. Apply the properties of determinants and calculate:

$$A = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \quad B = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 4 & 16 & 36 & 64 \\ 8 & 64 & 216 & 512 \end{vmatrix} \quad C = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 0 & 3 & 4 & 1 \\ -1 & 2 & 2 & 5 \\ 2 & -2 & 1 & -3 \end{vmatrix}$$

$$A = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \xrightarrow[\substack{r_3-r_2 \\ r_4-r_3}]{r_2-r_1} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} = \xrightarrow[r_4-r_3]{r_3-r_2} \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix} = 0$$

$$B = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 4 & 16 & 36 & 64 \\ 8 & 64 & 216 & 512 \end{vmatrix} \xrightarrow[\substack{c_3-c_2 \\ c_4-c_3}]{c_2-c_1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 4 & 12 & 20 & 28 \\ 8 & 56 & 152 & 296 \end{vmatrix} =$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ 12 & 20 & 28 \\ 56 & 152 & 296 \end{vmatrix} \xrightarrow[c_3-c_2]{c_2-c_1} = 2\begin{vmatrix} 1 & 0 & 0 \\ 12 & 8 & 8 \\ 56 & 96 & 144 \end{vmatrix} =$$

$$= 2\begin{vmatrix} 8 & 8 \\ 96 & 124 \end{vmatrix} = 2\cdot 8\cdot 8\begin{vmatrix} 1 & 1 \\ 12 & 18 \end{vmatrix} = 128\,(18-12) = 768$$

$$C = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 0 & 3 & 4 & 1 \\ -1 & 2 & 2 & 5 \\ 2 & -2 & 1 & -3 \end{vmatrix} \xrightarrow[r_4-2r_1]{r_3+r_1} = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 8 \\ 0 & -6 & -3 & -9 \end{vmatrix} =$$

$$= \begin{vmatrix} 3 & 4 & 1 \\ 4 & 4 & 8 \\ -6 & -3 & -9 \end{vmatrix} = -3\cdot 4\begin{vmatrix} 3 & 4 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} =$$

$$= -3\cdot 4\left[9+16+1-(2+12+6)\right] = -72$$

2.11 Solution of Simultaneous Linear Equations

In mathematics, **simultaneous equations** are a set of equations containing multiple variables. This set is often referred to as a **system of equations**. A solution to a system of equations is a particular specification of the values of all variables that simultaneously satisfies all of the equations. To find a solution, the solver needs to use the provided equations to find the exact value of each variable. Generally, the solver uses either a graphical method, the matrix method, the substitution method, or the elimination method. This is a set of linear equations, also known as a linear system of equations: $2x + y = 8$ and $x + y = 6$

Solving this involves subtracting $x + y = 6$ from $2x + y = 8$ (using the elimination method) to remove the y-variable, then simplifying the resulting equation to find the value of x, then substituting the x-value into either equation to find y.

The solution of this system is: $x = 2$ and $y = 4$

which can also be written as an ordered pair (2, 4), representing on a graph the coordinates of the point of intersection of the two lines represented by the equations.

An example of solving 3 linear equations can be expressed as

$a_1x + b_1y + c_1y + d_1 = 0$

$a_2x + b_2y + c_2y + d_2 = 0$

$a_3x + b_3y + c_3y + d_3 = 0$

Using determinants this is solved by the following relationship

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example:

Solve the equation.

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

This equation is rewritten as

$5x-6y+4z-15=0$

$7x+4y-3z-19=0$

$2x + y + 6z - 46 = 0$

Expressing this in determinant form

$$\frac{x}{\begin{vmatrix} -15 & -6 & 4 \\ -19 & 4 & -3 \\ -46 & 1 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 5 & -15 & 4 \\ 7 & -19 & -3 \\ 2 & -46 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 & -15 \\ 7 & 4 & -19 \\ 2 & 1 & -46 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}}$$

On evaluating the denominators.

$(x /-1257) = (y /-1676) = (z / -2514) = (-1 /419)$

Dividing each denominator by 419 results in

$(x /—3) = (y /-4) = (z / -6) = -1$

These results in

$x = 3$: $y = 4$: $z = 6$

The expansion of the determinant is simplified by using the row or column with zero elements if anyas the relevant element-co-factor is zero..

Apart from the above solution there are three methods to show that how simultaneous linear equations can be solved

Three methods:

1) Gauss –Jordan Elimination /Do little Method

In linear algebra, **Gauss–Jordan elimination** is an algorithm for getting matrices in reduced row echelon form using elementary row operations. It is a variation of Gaussian elimination. Gaussian elimination places zeros below each pivot in the matrix, starting with the top row and working downwards. Matrices containing zeros below each pivot are said to be in row echelon form. Gauss–Jordan elimination goes a step further by placing zeros above and below each pivot; such matrices are said to be in reduced row echelon form. Every matrix has a reduced row echelon form, and Gauss–Jordan elimination is guaranteed to find it.

It is named after Carl Friedrich Gauss and Wilhelm Jordan because it is a variation of Gaussian elimination as Jordan described in 1887. However, the method also appears in an article by Clasen published in the same year. Jordan and Clasen probably discovered Gauss–Jordan elimination independently.

Gauss-Jordan reduction

Step 1: Form the augmented matrix corresponding to the system of linear equations.

Step 2: Transform the augmented matrix to the matrix in reduced row echelon form via elementary row operations.

Step 3: Solve the linear system corresponding to the matrix in reduced row echelon form. The solution(s) are also for the system of linear equations in step 1.

Example:

Solve the following linear system:

$$\begin{aligned} x_1 + x_2 + 2x_3 - 5x_4 &= 3 \\ 2x_1 + 5x_2 - x_3 - 9x_4 &= -3 \\ 2x_1 + x_2 - x_3 + 3x_4 &= -11 \\ x_1 - 3x_2 + 2x_3 + 7x_4 &= -5 \end{aligned}$$

Solution:

The Gauss-Jordan reduction is as follows:

Step 1:

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{bmatrix}.$$

Step 2:

After elementary row operations, the matrix in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 3:

The linear system corresponding to the matrix in reduced row echelon form is

$$\begin{aligned} x_1 \quad\quad + 2x_4 &= -5 \\ x_2 \quad - 3x_4 &= 2 \\ x_3 - 2x_4 &= 3 \end{aligned}$$

The solutions are

$$x_1 = -5 - 2t,\ x_2 = 2 + 3t,\ x_3 = 3 + 2t,\ x_4 = t,\ t \in R$$

$$\Leftrightarrow \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5-2t \\ 2+3t \\ 3+2t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \\ 1 \end{bmatrix} t$$

Number of solutions of a system of linear equations:

For any system of linear equations, precisely one of the following is true.

I. The system has exactly one solution.

II. The system has an infinite number of solutions.

III. The system has no solution.

Note: The linear system with at least one solution is called consistent and the linear system with no solution is called inconsistent.

2) Cramer's Rule

Cramer's rule is a method of solving a system of linear equations through the use of determinants. Cramer's rule is given by the equation

$$x_i = \frac{|A_i|}{|A|}$$

where x_i is the i^{th} endogenous variable in a system of equations, $|A|$ is the determinant of the original A matrix as discussed in the previous section, and $|A_i|$ is the determinant a **special** matrix formed as part of Cramer's rule.

To use Cramer's rule, two (or more) linear equations are arranged in the matrix form $Ax = d$. For a two equation model:

$$\begin{array}{ccccc} A & & x & = & d \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{array}$$

A is the matrix corresponding to the number of equations in a system (here, two equations), and the number of endogenous variables in the system (here 2 variables). Remember that the matrix must be square, so the number of equations must equal the same number of endogenous variables. Position x has one column and corresponds to the number of endogenous variables in the system. Finally, position d contains the exogenous terms of each linear equation.

Note: The determinant for a matrix must not equal 0 ($|A| \neq 0$). If $|A| = 0$ then there is no solution, or there are infinite solutions (from dividing by zero). Therefore, $|A| \neq 0$. When $A \neq 0$, then a unique solution exists.

Applying Cramer's Rule in a 2x2 example

Using Cramer's rule to solve for the unknowns in the following linear equations:

$$2x_1 + 6x_2 = 22$$

$$-x_1 + 5x_2 = 53$$

Then,

$$\begin{array}{ccc} A & x & = \quad d \\ \begin{bmatrix} 2 & 6 \\ -1 & 5 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} 22 \\ 53 \end{bmatrix} \end{array}$$

The primary determinant $|A| = \begin{vmatrix} 2 & 6 \\ -1 & 5 \end{vmatrix} = 2\,(5) - (-1)\,6 = 16$

We need to construct $x_i = \dfrac{|A_i|}{|A|}$, for i=1 and for i=2.

The first special determinant A_1 is found by replacing the first column of the primary matrix with the constant 'd' column. The new special matrix A_1 now appears as:

$$A_1 = \begin{bmatrix} 22 & 6 \\ 53 & 5 \end{bmatrix}$$

and solved as a regular matrix determinant,

$$|A_1| = 22\,(5) - 53\,(6) = -208$$

Likewise, the same procedure is done to find the second special determinant A_2,

$$A_2 = \begin{bmatrix} 2 & 22 \\ -1 & 53 \end{bmatrix}$$

$$|A_2| = 2\,(53) - (-1)\,(22) = 128$$

We have now determined:

$$|A| = 16$$
$$|A_1| = -208$$
$$|A_2| = 128$$

Using:

$$x_i = \frac{|A_i|}{|A|}$$

We get,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-208}{16} = -13$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{128}{16} = 8$$

Applying Cramer's Rule in a 3x3 example

Using Cramer's Rule to solve for the unknowns in three linear equations:

$$5x_1 - 2x_2 + 3x_3 = 16$$

$$2x_1 + 3x_2 - 5x_3 = 2$$

$$4x_1 - 5x_2 + 6x_3 = 7$$

Then,

$$\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 7 \end{bmatrix}$$

The primary determinant $|A| = \begin{vmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{vmatrix} = 5(18 - 25) + 2(12 + 20) + 3(-10 - 12) = -37$

The three special determinants are:

$$|A_1| = \begin{vmatrix} 16 & -2 & 3 \\ 2 & 3 & -5 \\ 7 & -5 & 6 \end{vmatrix} = 16(18 - 25) + 2(12 + 35) + 3(-10 - 21) = -111$$

$$|A_2| = \begin{vmatrix} 5 & 16 & 3 \\ 2 & 2 & -5 \\ 4 & 7 & 6 \end{vmatrix} = 5(12 + 35) - 16(12 + 20) + 3(14 - 8) = -259$$

$$|A_3| = \begin{vmatrix} 5 & -2 & 16 \\ 2 & 3 & 2 \\ 4 & -5 & 7 \end{vmatrix} = 5(21 + 10) + 2(14 - 8) + 16(-10 - 12) = -185$$

Applying Cramer's Rule:

$$x_1 = \frac{|A_1|}{A} = \frac{-111}{-37} = 3$$

$$x_2 = \frac{|A_2|}{A} = \frac{-259}{-37} = 7$$

$$x_3 = \frac{|A_3|}{A} = \frac{-185}{-37} = 5$$

3) Matrix Inverse Method:

The 'transpose' of a matrix is often referenced, but what does is mean? It sure has an algebraic interpretation but we do not know if that could be expressed in just a few words. Anyway,we define it by taking examples to find out what the pattern is. Below is a 2x2 matrix like it is used in complex multiplication. The transpose of a square matrix can be considered a mirrored version of it: mirrored over the main diagonal. That is the diagonal with the s on it.

Example,

Find transpose of the matrix A =

And verify that $(A')' = A$

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 5 & -3 \end{pmatrix}$$

Solution. From the definition of the transpose of a matrix, A' is obtained by interchanging the rows and columns of the matrix A. Thus

$$A' = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & -3 \end{pmatrix}$$

Further , taking the transpose of the matrix A', we have

$$(A')' = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 5 & -3 \end{pmatrix} = A$$

Hence $(A')' = A$

The following transpose properties are observed in matrices

$(ABC)^T = C^T B^T A^T$

$(ABC^T)^T = (C^T)^T B^T A^T = CB^T A^T$

The following inverse properties are observed in matrices

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

$(ABC^{-1})^{-1} = (C^{-1})^{-1} B^{-1} A^{-1} = CB^{-1} A^{-1}$

$A^{-1}A = AA^{-1} = I = 1$

Since matrix division is not defined, it is impossible to divide a matrix expression by a given matrix.

Finding out Adjoint of a Matrix :

Adjoint or Adjugate Matrix of a square matrix is the transpose of the matrix formed by the cofactors of elements of determinant |A|.
To calculate adjoint of matrix we have to follow the procedure
a) Calculate Minor for each element of the matrix.
b) Form Cofactor matrix from the minors calculated.
c)Finding cofactors from Minors
d) Form Adjoint from cofactor matrix.

For example we will use a matrix

$$A = \begin{pmatrix} A11 & a12 & a13 \\ A21 & a22 & a23 \\ A31 & a32 & a33 \end{pmatrix}$$

Step 1: Calculate Minor for each element.

To calculate the minor for an element we have to use the elements that do not fall in the same row and column of the minor element.

$$\text{Minor of } A_{11} = M_{11} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} = A_{22}\, x A_{33} - A_{32}\, x A_{23}$$

$$\text{Minor of } A_{12} = M_{12} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix} = A_{21}\, x A_{33} - A_{31}\, x A_{23}$$

$$\text{Minor of } A_{13} = M_{13} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} = A_{21}\, x A_{32} - a_{31}\, x A_{22}$$

$$\text{Minor of } A_{21} = M_{21} = \begin{pmatrix} A_{11} & A_{12} & A13 \\ A_{21} & A_{22} & A23 \\ A_{31} & A_{32} & A33 \end{pmatrix} = \begin{pmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{pmatrix} = A_{12}\, x A_{33} - A_{32}\, x A_{13}$$

Similarly

$M_{22} = A_{11}\, xA_{33} - A_{31}\, xa_{13}$ $\quad M_{23} = A_{11}\, xA_{32} - A_{31}\, xA_{12}$

$M_{31} = A_{12}\, xA_{23} - A_{22}\, xA_{13}$ $\quad M32 = A_{11}\, xA_{23} - A_{21}\, xA_{13}$

Step 2: Form a matrix with the minors calculated.

$$\text{Matrix of Minors} = \begin{pmatrix} M_{11} & M_{12} & M_{31} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

Step 3: Finding the cofactor from Minors:

Cofactor: A signed minor is called cofactor.

The cofactor of the element in the i^{th} row, j^{th} column is denoted by $\mathbf{C_{ij}}$

$\mathbf{C_{ij}} = (-1)^{i+j} M_{ij}$

$$\text{Matrix of Cofactors} = \begin{pmatrix} (-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & (-!)^{1+3}M_{13} \\ (-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & (-1)^{2+3}M_{23} \\ (-1)^{3+1}M_{31} & (-1)^{3+2}M_{32} & (-1)^{3+3}M_{33} \end{pmatrix}$$

So,

$$\begin{pmatrix} C_{11} & C_{12} & C_{31} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} M_{11} & -M_{12} & M_{31} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}$$

Step 4: Calculate adjoint of matrix:

To calculate adjoint of matrix, just put the elements in rows to columns in the cofactor matrix. i.e convert the elements in first row to first column, second row to second column, third row to third column.

$$\text{Adjoint of Matrix} = \begin{pmatrix} C_{11} & C_{12} & C_{31} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

So we can say that,

$$A^{-1} = \frac{1}{\det A}(\text{adjoint of } A) \quad \text{or} \quad A^{-1} = \frac{1}{\det A}(\text{cofactor matrix of } A)^T$$

Example: The following steps result in A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

The cofactor matrix for A is $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$, so the adjoint is $\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$. Since det A = 22, we get

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

2.12 Application of Matrices to Business Problems

We can measure the expansion of the world by matrices cause in magnetic fields vectors can be stretched up to a certain limit which are the eigen values. It enables students to interact with numbers. Matrices are used basically to solve linear equations....... Equations are generally solved to get value of unknown variables.......Variable values are calculated (or assumed) to know all working. varied applications. A few are: 1. Which group of customers are more valuable (profitable) to the business 2. How vary the business results are from the average 4. Predict the business future.

The thing about "business applications" is that they can usually be run on equipment over five years old. They simply do not demand much in the way of features or power.

Example

1) Ram Shyam and Mohan purchased biscuits of different brands P, Q and R . Ram purchased 10 packets of P, 7 packets of Q and 3 packets of R. Shyam purchased 4 packets of P, 8 packets of Q and 10 packets of R.Mohan purchased 4 packets of P, 7 packets of Q and 8 packets of R. If brand P costs Rs .4, Q costs Rs. 5 and R costs Rs. 6 each , then using matrix operations find the amount of money spent by these persons individually.

Solution.

Let Q be the matrix denoting the quantity of each brand of biscuit bought by P, Q and R and let C be the matrix showing the cost of each brand of biscuit

$$C = \begin{pmatrix} \text{Ram} \\ \text{Shaym} \\ \text{Mohan} \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{pmatrix} \end{array}$$

$$\begin{array}{c} P \\ Q \\ R \end{array} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Since number of columns of first matrix should be equal to the number of rows of the second matrix for multiplication to be possible, the above matrices shall be multiplied in the following order

$$Q^* \, C = \begin{pmatrix} 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{pmatrix} * \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 93 \\ 116 \\ 99 \end{bmatrix}$$

Amount spent by Ram, Shyam and Mohan is Rs 99, Rs 116 and Rs 99 respectively

2.13 Summary

Calculus has two main parts: differential calculus and integral calculus. Differential calculus studies the derivative and integral calculus studies (surprise!) the integral. The derivative and integral are linked in that way that they are both defined via the concept of the limit: they are inverse operations of each other (a fact sometimes known as the fundamental theorem of calculus): and they are both fundamental to much of modern science as we know it.

Derivatives can be used to help you graph functions. First, they give you the slope of the graph at a point, which is useful. Second, the points where the slope of the graph is horizontal ($f'(x) = 0$) are particularly important, because these are the only points at which a relative minimum or maximum can occur (in a differentiable function).

Methods of computing determinants:

(i) Expand along a row or down a column

(ii) Simplify using elementary row (or column) operations to get to upper triangular form.

(iii) A mixture of (i) and (ii), i.e. use some row (or column) operations to simplify before expanding what's left.

Methods of computing A"1:

(i) Standard formula in 2 × 2 case;

(ii) Adjugate matrix method: A"1 = (detA)"1adjA.

Too hard for large n, but easy for n = 2 (gives formula referred to in (i)), and still possible for n = 3.

(iii) Elementary row operations on augmented matrix (Gaussian elimination). Best general method for large n.

Methods of solving AX = B:

(i) Inverse matrix method: X = A"1B.

Depends on knowing A-1 — good way if A-1 is already known or is easy tofind, and especially good if there are several B's with only one A.

(ii) Elementary row operations (or Gaussian elimination). Best general method, especially for large n.

(iii) Cramer's rule: xi = detBi/ detA, where Bi is formed from A by substituting B for i'th column.

2.14 Key Words

- **Function :** A function f is a relation from a non empty set A into a non empty set B such that domain of f is A and no two ordered pairs in f have the same first component. Equivalently, a function f from a set A into set B is a relation from A into B if for each a"B such that
- **First derivative**: If the first derivative is positive, the function is increasing; if the first derivative is negative, the function is decreasing.If f '(a) = 0 then a is called a *critical point* of f.
- **Second derivative**: If f ''(a) = 0 then a is called an *inflection point* of f.
- **Explicit function** is a function in which the dependent variable has been given "explicitly"
- **Differentiation:** is defined as a method to compute the rate at which a dependent output y changes with respect to the change in the independent input x.
- **Crammers Rule:** The linear system $AX = B$ has a unique solution if and only if A is invertible. In this case, the solution is given by the so-called **Cramer's formulas**
- **Co-Factor of Determinant:-** Associated with each element is a sign (+ or -) this is established by counting the number of row and column steps to get from the first element and if the steps are 0 or even the sign is (+) and if the steps are odd the sign is (-).

2.15 Self Assessment Test

1 There are three dealers in a market who sales Radios, T.V and CD players . A sells weekly 14 radios, 2 TV and CD players. B sells weekly 10 radios, 3 TV and 10 CD players. C sells weekly 9 radios, 5 TV and 9 CD players. If the profit per radio is Rs. 50, on TV is Rs. 20 and on CD players is Rs. 30. Calculate their individual profits.

2 A company has two plants P1 and P2, P1 is capable of producing 5 units of A, 10 units of B and 3 units of C per hour of operation. P2 is capable of producing 5 units of A, 6 units of B and 6 units of C per hour of operation. Using matrix algebra, determine the total number of units of A, B and C produced if P1 is operated for 10 hrs and P2 is operated for 5 hours.

3 A man invested Rs. 30000 into three different investments. The rates of interest is 2%, 3% and 4% per annum resp. The total annual income is Rs. 1000. If the income from the first and second investments is Rs. 50 more than the income from third, find the amount of each investment, by using matrix algebra.

4 Vitamin A and B are found in two different foods F1 and F2. One unit of food F1 contains 3 units of vit A and 8 units of vitamin B. One unit of food F2 contains 4 units of vitamin A and 6 units of vitamin B. The daily requirements of vitamin A and B is 24 and 48 units resp. Calculate the optimum mixture of food F1 and F2 to be taken to meet the daily requirements of vitamin A and B.

5 Show that $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix} = 0$

6 Solve the following system of linear equations using the method of matrix reduction

$x + y + z = 6$

$x + 2y + 3z = 14$

$-x + y - z = -2$

7 For square matrices A and B , expand (A + B) (A – B) and (A – B) (A + B). When will these be equal?

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Unit - 3 : Correlation and Simple Regression

Unit Structure:

3.0 Objectives

After studying this unit, you will be able to

- Define Correlation and Regression.
- Differentiate between Correlation and Regression.
- Find the equation of the lines of Regression.
- Find out how much a variable changes corresponding to the average amount of change in the other variables.

3.1 Introduction

From statistics point of view Correlation and Regression are the tools to measure the amount of similarity and variations of two or more variables. These are used for studying and measuring the extent of relationship between two or more variables. Generally, there are situations where there is a certain degree of association but not of a fixed type between variables which may occurs in pair or groups. For example, consider the heights and weights of 100 students in a class. We want to measure the heights and weights of all the students. We observe that if the height and weight of a student is represented by x and y respectively, then there corresponds a pair of values (x_i, y_i) of the variables x and y for each measurement. Such a distribution in which each individual of the set is made up of two values is called bivariate distribution.

If the changes in the value of one variable appear to be related to the changes in the value of another variable, then the two variables are said to be correlated. For example, pressure and volume of a gas are correlated because with the increase in pressure there is decrease in volume and the vice versa. Other similarly placed situations are:

- The amount of yield of grain per acre and the amount of fertilizer per acre.
- The sales revenue and advertising expenditure of a firm in number of years.
- The heights of the husband and the wives at the time of marriage.

In a bivariate distribution, we intend to find a relationship (if it exist) between two variables under study. Francis Galton used the term 'Regression' in the latter part of nineteenth century. Regression is used to find out some sort of functional relationship between two or more variables. The average relationship between the correlated variables is estimated by Regression. The main objective of correlation is to study the relation (if it exists) between two or more variables which are so related that a change in one variable is accompanied by changes in other variable. Regression, on the other hand, aims at estimation of unknown values of one variable from the known values of other variable.

3.2 Correlation

We define Correlation as a measure of relation between two variables. In other words, to describe and understand the association between two continuous variables (interval or ratio data), we compute Correlation. Thus two variables are said to be correlated if an increase (or decrease) in one variable is accompanied by an increase or decrease (decrease or increase) in the other variable.

Types of Correlation

Correlation is classified in several different ways:

1. Positive or Negative correlation
2. Linear and Non-linear correlation
3. Simple, Multiple and Partial correlation
4. Spurious correlation

Positive Correlation

Two variables are said to be positively correlated, if an increase in the value of one is accompanied by an increase in the value of other or a decrease in the value of one is accompanied by a decrease in the value of other i.e. the value of two variables deviate in the same direction. For example, paired variables like supply and demand of commodities, household income and expenditure, price and supply of commodities is positively correlated.

Examples:

X :	10	12	14	16	18
Y:	15	20	25	30	35

X :	80	75	70	60	50
Y:	6	5	4	2	1

Negative Correlation

Two variables are said to be negatively correlated, if an increase in the value of one is accompanied by a decrease in the value of other or a decrease in the value of one is accompanied by an increase in the value of other. For example, paired variables like pressure and volume of a gas, current and resistance (voltage being constant), demand and price of commodities are negatively correlated.

Examples:

X :	10	12	14	16	18
Y :	15	12	10	8	6

X :	80	75	70	60	50
Y :	60	70	75	80	90

Linear Correlation

The correlation between two variables is said to be linear, if there exists a relationship of the form:

$$y = a + bx$$

where, a and b are real numbers.

In a linear correlation the amount of change in one variable tends to bear constant ratio to the amount of change in the other variable. The graph of linear correlation is a straight line.

Example:

X :	10	12	14	16	18
Y :	15	20	25	30	35

Non Linear Correlation

The correlation between two variables is said to be non - linear, if there exists a relationship of the form of polynomial of order more than one, for example:

$$y = a + bx + cx^2$$

where, a , b and c are real numbers.

In the non - linear correlation, a change of one unit in one variable does not correspond to same amount of change in the other variable. The graph of non - linear correlation is not a straight line.

Example:

X :	10	12	14	16	18
Y :	15	20	22	30	40

Simple Correlation

In simple correlation the number of variables studied/ included is two. All the above mentioned correlations are simple; including positive, negative, linear and non-linear.

Multiple Correlation

When three or more variables are studied simultaneously it is a case of multiple correlation. For example, when we study the relationship between the yield of rice per acre, amount of rainfall and the amount of fertilizers used, it is a problem of multiple correlation.

Partial Correlation

In partial correlation we recognize more than two variables, but consider only two variables to be influencing each other the effect of other influencing variables being kept constant. For example, in the rice problem taken above if we limit our correlation analysis of yield and rainfall to periods when a certain average daily temperature existed it becomes a problem relating to partial correlation only.

Spurious correlation

Given data regarding any two variables, it is possible that on calculation of r, one may say that statistically the two variables are correlated, but if there is no justifiable, logical explanation for correlation to exists, then such statistical correlation is termed as **spurious correlation**. In a spurious correlation two variables have no direct causal connection, yet it may be wrongly inferred that they do, due to either coincidence or the presence of a certain third, unseen factor (referred to as a *"confounding factor"* or "*lurking variable*"). Suppose there is found to be a correlation between A and B. Aside from coincidence, there are three possible relationships:

A causes B,

B causes A,

OR

C causes both A and B.

In the last case there is a spurious correlation between A and B. In a regression model where A is regressed on B but C is actually the true casual factor for A, this misleading choice of independent variable (B instead of C) is called specification error.

Scatter Diagram

Suppose we are given two paired variable values - one is related to the heights of the students while the other is related to the weights of the students. They are plotted on the coordinate plane using x-axis to represent heights and y-axis to represent weights. Then each paired observation shall have one point on the graph. The graphical representation of dots so obtained is called Scatter Diagram. The different graphs shown below illustrate the different type of correlations.

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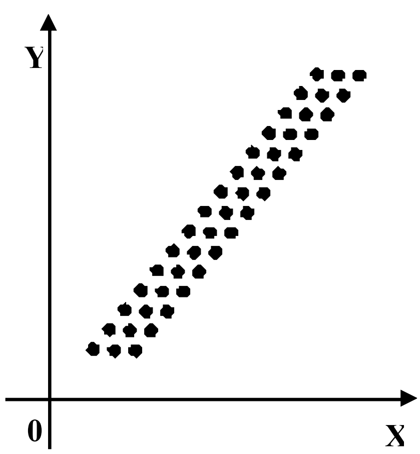
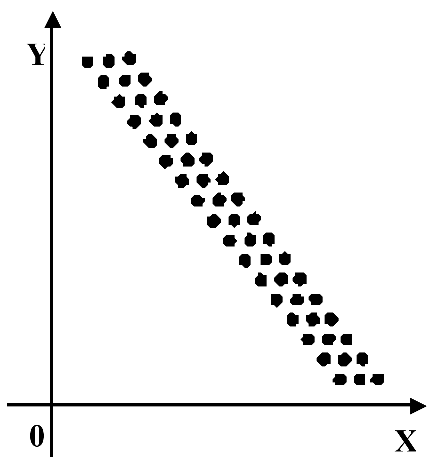
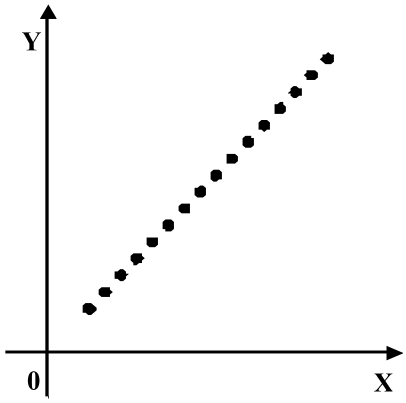


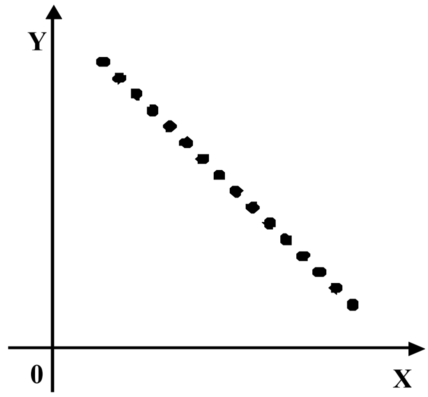
High degree positive correlation



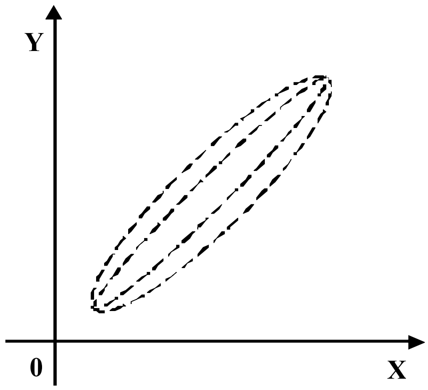
High degree negative Correlation



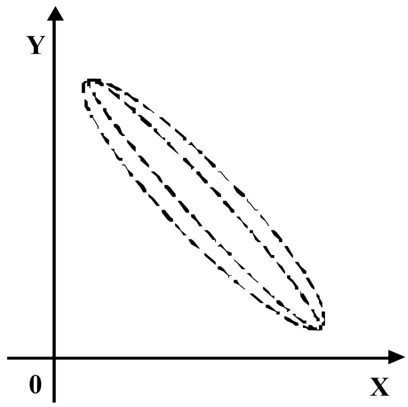
Perfect positive correlation



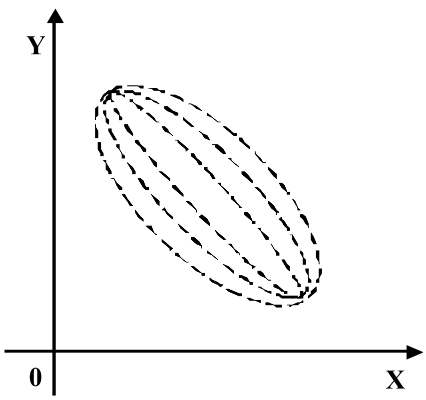
Perfect negative correlation



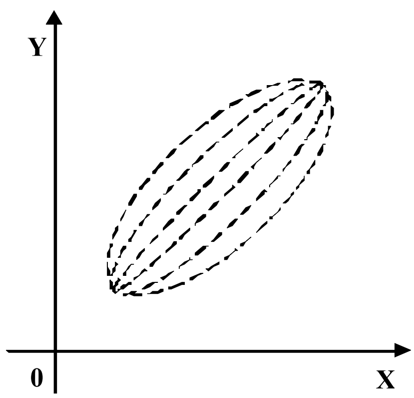
Moderate degree +ve correlation



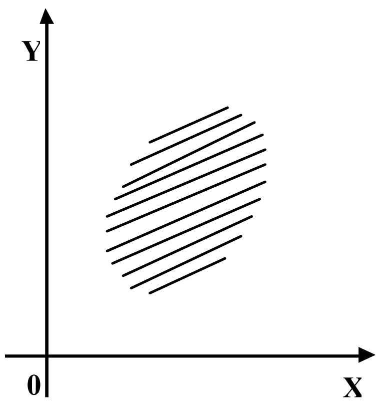
Moderate degree -ve correlation



Low degree -ve correlation



Low degree +ve correlation



No correlation

3.3 Correlation Coefficient

The Coefficient of Correlation '*r*' is a measure of the degree of linear relationship between two variables, say, ***x*** and ***y*** i.e. it measures the degree of association between the two values of related variables given in the data set.

(i) The coefficient of correlation '*r*' (also known as Karl Pearson's Coefficient of Correlation) is given by:

$$r = \frac{n\sum xy - \sum x.\sum y}{\sqrt{\{n\sum x^2 - (\sum x)^2\}.\{n\sum y^2 - (\sum y)^2\}}}$$

(ii) If the mean of variables ***x*** and ***y***, are denoted by $\bar{x}$ and $\bar{y}$, than we can apply the following formula to calculate the coefficient of correlation '*r*':

$$r = \frac{\sum dx.dy}{\sqrt{\sum dx^2 . \sum dy^2}} \qquad \text{where } dx = x - \bar{x}, \quad dy = y - \bar{y}$$

difference of values from its arithmetic mean

(iii) If the difference from any constants '***A***' or '***B***' (other than arithmetic mean) is taken the following formula can be used to calculate the coefficient of correlation '*r*'.

$$r = \frac{n\sum dx.dy - \sum dx.\sum dy}{\sqrt{\{n\sum dx^2 - (\sum dx)^2\}\{n\sum dy^2 - (\sum dy)^2\}}} \qquad \text{where } dx = x - A, \quad dy = y - B$$

The value of coefficient of correlation 'r' lies between - 1 and + 1. If for a given two sets or data:

- ***r*** = +1, the two sets or data are said to be perfect positively correlated.
- ***r*** = - 1, the two sets or data are said to be perfect negatively correlated.
- ***r*** = 0, the two sets or data are said to be uncorrelated, that is, there is absence of any linear relationship between the variables. However, there may exist some other form of relationship between them.
- ***r*** = 0.8, the two sets or data are said to be strongly correlated.
- ***r*** = 0.2, the two sets or data are said to be weakly correlated.

Properties of Coefficient of Correlation '*r*'

1. The value of '*r*' is independent of the origin of reference and the scale of reference i.e. '*r*' is not affected by addition or subtraction of a constant to the values of either or both variables. It is also, unaffected by the multiplication or division of the values of either or both variables, by a constant.
2. The value of '*r*' is free from any of the units. In other words, '*r*' will have a definite meaning, whether the units of ***x*** and ***y*** variables are comparable or not.
3. The coefficient of correlation is symmetrical in two variables.
4. The value of '*r*' lies between + 1 and - 1.

Example1: Calculate the coefficient of correlation between X and Y series:

X :	1	3	5	7	8	10
Y :	8	12	15	17	18	20

Solution:

Table for Calculation

X	Y	X^2	Y^2	XY
1	8	1	64	8
3	12	9	144	36
5	15	25	225	75
7	17	49	289	119
8	18	64	324	144
10	20	100	400	200
$\sum X = 34$	$\sum Y = 90$	$\sum X^2 = 248$	$\sum Y^2 = 1446$	$\sum XY = 582$

The coefficient of correlation '*r*' is given by:

$$r = \frac{n\sum XY - \sum X . \sum Y}{\sqrt{\{n\sum X^2 - (\sum X)^2\}.\{n\sum Y^2 - (\sum Y)^2\}}} \qquad \text{- - - (1)}$$

Substituting the values from the table in the equation (1), we obtain:

$$r = \frac{6 . (582) - (34) . (90)}{\sqrt{\{6 . (248) - (34)^2\}.\{6 . (1446) - (90)^2\}}}$$

$$r = \frac{(3492 - 3060)}{\sqrt{(332) . (576)}}$$

$$r = \frac{432}{437.3} = 0.988$$

Example 2: Calculate the coefficient of correlation between the height of father and son from the following data:

Height of Father (in Inches) (x) :	64	65	66	67	68	69	70
Height of Son (in Inches) (x) :	66	67	65	68	70	68	72

Solution:

Height of Father (in inches) x	Height of Son (in inches) y	dx (x – 67)	dy (y – 68)	dx^2	dy^2	dx.dy
64	66	-3	-2	9	4	+6
65	67	-2	-1	4	1	+2
66	65	-1	-3	1	9	+3
67	68	0	0	0	0	0
68	70	+1	+2	1	4	+2
69	68	+2	0	4	0	0
70	72	+3	+4	9	16	+12
$\sum x = 469$	$\sum y = 476$	$\sum dx = 0$	$\sum dy = 0$	$\sum dx^2 = 28$	$\sum dy^2 = 34$	$\sum dx.dy = 25$

$$where\ \bar{x} = 67$$
$$\bar{y} = 68$$
$$dx = x - \bar{x}$$
$$dy = y - \bar{y}$$

The coefficient of correlation $'r'$ is given by:

$$r = \frac{\sum dx\ .dy}{\sqrt{\sum dx^2 . \sum dy^2}} \quad ---(1)$$

Substituting the values from the Table in equation (1), we obtain

$$r = \frac{25}{\sqrt{28\ .\ 43}}$$

$$r = \frac{25}{30.85} = 0.81$$

Spearman's Rank Correlation

Ranked data are defined as data which are arranged in numerical order, usually from largest to smallest and numbered 1, 2, 3, . . . The correlation of such type of data is known as Rank Correlation, as here only ranks are considered and denoted by 'R'.

The coefficient of Rank correlation R is given by:

$$\mathbf{R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}}$$

where, '***n***' is the total number of individuals and '***d'*** is the difference in ranks.

where, '***n***' is the total number of individuals and '***d'*** is the difference in ranks.

Properties of Rank Correlation Coefficient:

- Rank Correlation Coefficient lies between +1 and - 1, including both the values.
- If d_j stands for the difference in the ranks of the jth individual and if $d_j = 0$ for all individuals, then Rank correlation coefficient 'R' = 1.

Example 3: The marks obtained by 8 students in Physics and Chemistry are given in the following Table:

Marks in Physics :	52	54	67	82	98	90	69	76
Marks in Chemistry :	11	7	23	36	56	37	12	25

Calculate the Rank Correlation Coefficient.

Solution:

First change the given values into ranks for each series then after giving ranks to the students in Physics and Chemistry and calculating the difference '***d'*** in ranks, we obtain the values shown in following Table:

Marks in Physics (x)	Marks in Chemistry (y)	Rank in Physics	Rank in Chemistry	Difference in Rank (d)	d^2
52	11	8	7	+1	1
54	7	7	8	-1	1
67	23	6	5	+1	1
82	36	3	3	0	0
98	56	1	1	0	0
90	37	2	2	0	0
69	12	5	6	-1	1
76	25	4	4	0	0
					$\sum d^2 = 4$

The Rank Correlation Coefficient is given by:

$$\mathbf{R = 1 - \frac{6\sum d^2}{n(n^2 - 1)}}$$

$$R = 1 - \frac{6\,.\,4}{8\,(64 - 1)}$$

$$= 1 - \frac{24}{504}$$

$$= 1 - 0.04761$$

$$= 0.95238$$

Example 4: Ten competitors in a beauty contest are ranked by three judges in the following order:

1st judge	1	6	5	10	3	2	4	9	7	8
2nd judge	3	5	8	4	7	10	2	1	6	9
3rd judge	6	4	9	8	1	2	3	10	5	7

Use the Rank Correlation Coefficient to determine which pair of judges has the nearest approach to appreciation of beauty.

Solution:

In order to find out which pair of judges has the nearest approach to appreciation of beauty, we compare Rank Correlation between the judgments of:

(i) 1st judge and 2nd judge

(ii) 2nd judge and 3rd judge

(iii) 1st judge and 3rd judge

Rank by 1st judge R_1	**Rank by 2nd judge R_2**	**Rank by 3rd judge R_3**	$(R_1 - R_2)^2$	$(R_2 - R_3)^2$	$(R_1 - R_3)^2$
1	3	6	4	9	25
6	5	4	1	1	4
5	8	9	9	1	16
10	4	8	36	16	4
3	7	1	16	36	4
2	10	2	64	64	0
4	2	3	4	1	1
9	1	10	64	81	1
7	6	5	1	1	4
8	9	7	1	4	1
N = 60	N = 60	N = 60	$\sum d^2 = 200$	$\sum d^2 = 214$	$\sum d^2 = 60$

The Rank Correlation Coefficient is given by:

$$\mathbf{R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}}$$

The Rank Correlation Coefficient between the judgments of 1st and 2nd judges::

$$R_{(\text{I and II})} = 1 - \frac{6 \,.\, 200}{10\ (100 - 1)}$$

$$= 1 - \frac{1200}{990}$$

$$= 1 - 1.212$$

$$= -\ 0.212$$

The Rank Correlation Coefficient between the judgments of 2nd and 3rd judges::

$$R_{(\text{II and III})} = 1 - \frac{6 \,.\, 214}{10\ (100 - 1)}$$

$$= 1 - 1.297$$

$$= -\ 0.297$$

The Rank Correlation Coefficient between the judgments of 1st and 3rd judges::

$$R_{(\text{I and III})} = 1 - \frac{6 \,.\, 60}{10\ (100 - 1)}$$

$$= 1 - \frac{360}{990}$$

$$= 1 - 0.3636$$

$$= 0.636$$

Since coefficient of correlation is maximum in the judgments of the first and third judges we conclude that they have the nearest approach to appreciation of beauty.

Equal Ranks If a tie occurs in the ranks of two individuals, they are assigned the average of ranks that they would have obtained. For example, if two individuals are tied up at the fifth rank, each may be ranked (5 +

6) / 2 = 5.5 and next is then ranked 7. Where equal ranks are assigned to some entries an adjustment in the above formula for calculating the rank coefficient of correlation is made.

The adjustment consists of adding $(m^3 - m)$ to the value of "d^2, where m stands for the number of items whose ranks are common. If there are more than one such group of items whose ranks are common, this value is added as many times the number of such groups.

Where equal ranks are assigned to some entries the coefficient of Rank correlation, R is given by:

$$R = 1 - \frac{6\left\{\sum d^2 + \frac{1}{12}(m^3 - m) + \frac{1}{12}(m^3 - m) + \dots\right\}}{n(n^2 - 1)}$$

Example 5: Obtain the rank correlation coefficient between the variables X and Y from the following pairs of observed values:

X:	50	55	65	50	55	60	50	65	70	75
Y:	110	110	115	125	140	115	130	120	115	160

Solution:

X	Rank X R_1	Y	Rank Y R_2	$(R_1 - R_2)^2$ d^2
50	2	110	1.5	0.25
55	4.5	110	1.5	9
65	7.5	115	4	12.25
50	2	125	7	25
55	4.5	140	9	20.25
60	6	115	4	4
50	2	130	8	36
65	7.5	120	6	2.25
70	9	115	4	25
75	10	160	10	0
				$\sum d^2 = 134$

In series X = 50 is repeated thrice (m = 3); 55 has been repeated twice (m = 2), 65 has been repeated twice (m = 2). In series Y; 110 has been repeated twice (m = 2) and 115 thrice (m = 3)

The Rank Correlation Coefficient is given by:

$$R = 1 - \frac{6\left\{\sum d^2 + \frac{1}{12}(m^3 - m) + \frac{1}{12}(m^3 - m) + \dots\right\}}{n(n^2 - 1)}$$

$$R = 1 - \frac{6\left\{134 + \frac{1}{12}(3^3 - 3) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(3^3 - 3) \dots\right\}}{10(10^2 - 1)}$$

$$= 1 - \frac{6\,\{134 + 2 + 0.5 + 0.5 + 0.5 + 2\}}{990}$$

$$= 1 - \frac{6\,(139.5)}{990} = 1 - \frac{837}{990} = 1 - .845 = 0.155$$

3.4 Simple Linear Regression

Sir Francis Galton introduced the term 'Regression' during the end of the 19th century. In today's world the term 'Regression' is used to measure the average relationship between the correlated variables. In other words, 'Regression' is the prediction of unknown values of one variable from the known values of other variable.

Regression Analysis is an important technique used to analyze the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps one to understand how the typical value of the dependent variable changes when any one of the independent variables is varied, while the other independent variables are held fixed. It helps to estimate the average value of the dependent variable from given values of independent variables. It is one of the most important statistical tools which is extensively used in almost all sciences - Natural, Social and Physical. It is specially used in business and economics to study the relationship between two or more variables that are related causally e.g. demand and supply, cost functions, production and consumption functions and so on.

Linear Regression involves a relationship between two variables only. There is a linear regression between the variables under study, if the regression curve is a polynomial of degree one. In other word, it means that if the set of observations for these two variables are plotted, a straight line is drawn through the scattergram.

The linear relationship between two variables x and y is represented by the following equation, which is a polynomial of degree one.

$$y = a + bx,$$

where, a and b are the constant determining the position of the line.

The slope of the line is denoted by b and intercept on the y-axis is given by a. This relationship is called Linear Regression of y on x. Here b is called regression coefficient of y on x. Similarly, the Linear Regression of x on y is given by equation:

$$x = c + dy,$$

where, c and d are constants. The constant d is called regression coefficient of x on y and the constant c is the intercept on x- axis.

It may be that all the points do not lie on the straight line equation which represents the linear relationship between two variables. The straight line equation shows the average relationship between the values of the two variables.

3.5 Estimating Linear Regression

The estimation or prediction of future production, consumption, prices, investments, sales, profits, income etc. are of very great Importance to business professionals. Similarly, population estimates and population projections, GNP, Revenue and Expenditure etc. are indispensable for economists and efficient planning of an economy. Regression estimates the changes in the value of dependent variables with respect to some values of independent variable. These estimation are done by the help of regression lines which are polynomials of degree one, having two variables.

Equation of Regression Lines

The straight line depicting the relationship between two variables is called Line of Regression or Regression Line. A Line of Regression is the one which gives the best fit using Principal of Least Square.

In order to obtain the equation of regression line, we will be using the Method of Least Squares. By the help of this method, we will obtain the best fitting line to the set of points on the scatter diagram. Suppose, the equation of line of best fit of x, be

$$y = a + bx$$

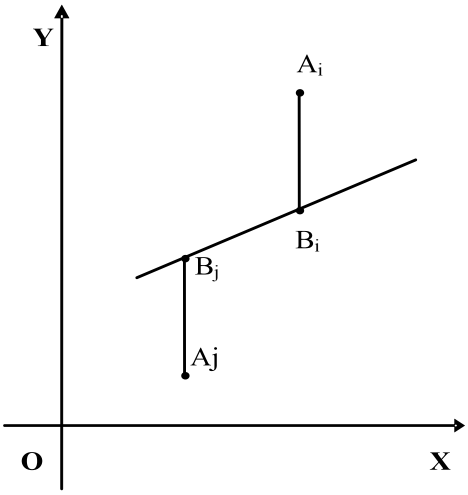
where, a and b are constants.

Let X and Y be the deviation from the respective means $\bar{x}$ and $\bar{y}$ i.e.

$$X = x - \bar{x} \quad , \quad Y = y - \bar{y}$$

Where , $\bar{x}$ and $\bar{y}$ are mean of x- series and y-series, respectively.

Let A_i be any point in graph and B_i be the corresponding point on the line (see the figure). Then, the distance between these points is represented by $A_i - B_i$.



If S be the sum of squares of such distances, then

$$S = \sum (A_i - B_i)^2$$

$$= \sum (Y_i - a - bX_i)^2$$

The principal of Least Squares says that sum of the squares of such distances be minimized. For this we have to choose those values of '***a***' and '***b***' for which this sum '***S***' is minimum. Each pair of '***a***' and '***b***' represents one regression line passing through the scattergram. We need to find that line for which this sum of squares is minimum.

Equation of line of best fit for Y on X is given by:

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

Similarly, we can obtain the equation of regression line of X on Y,

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_{xy}} (y - \bar{y})$$

$b_{yx} = r \frac{\sigma_y}{\sigma_x}$ and $b_{xy} = r \frac{\sigma_x}{\sigma_{xy}}$ are known as regression coefficient of **y on x** and **x on y**, respectively.

There are other variations of the above formula, which can also be used to find the regression equations, these are:

Solution of Normal Equations:

The regression equation of X on Y is expressed as:

$$X = a + b.Y$$

To determine the values of ***a*** and ***b***, the following two normal equations are to be solved simultaneously:

$$\sum X = N.\,a + b\,.\sum Y$$

$$\sum XY = a\,.\sum Y + b\,.\sum Y^2$$

*Solving these equations, values of **a** and **b** are obtained as follows:*

$$\boldsymbol{a} = \frac{\sum X - b.\sum Y}{\boldsymbol{N}}$$

$$\text{where,} \qquad \boldsymbol{b} = \frac{N.\sum XY - \sum X\,.\sum Y}{N.\sum Y^2 - (\sum Y)^2}$$

The regression equation of Y on X is expressed as:

$$Y = c + d.X$$

To determine the values of ***c*** and ***d***, the following two normal equations are to be solved simultaneously:

$$\sum Y = N.\,c + d\,.\sum X$$

$$\sum XY = c.\sum X + d\,.\sum X^2$$

*Similarly solving these equations, values of **c** and **d** are obtained as follows:*

$$\boldsymbol{c} = \frac{\sum Y - d\sum X}{\boldsymbol{N}}$$

$$\text{where,} \qquad \boldsymbol{d} = \frac{N.\sum XY - \sum X\,.\sum Y}{N.\sum X^2 - (\sum X)^2}$$

Example 6: Marks obtained by a student in Physics and Maths (out of 100) are given in the following Table:

Physics (x)	80	45	55	56	58	60	65	68	70	75	85
Maths (y)	82	56	50	48	60	62	64	65	70	74	90

Find the equations of lines of regression.

Solution:

Take, X = x - 65 and Y = y – 70

Table for Calculation

Physics (x)	Maths (y)	X = x - 65	Y = y - 70	X^2	Y^2	XY
80	82	15	12	225	144	180
45	56	-20	-14	400	196	280
55	50	-10	-20	100	400	200
56	48	-9	-22	81	484	198
58	60	-7	-10	49	100	70
60	62	-5	-8	25	64	40
65	64	0	-6	0	36	0
68	65	3	-5	9	25	-15
70	70	5	0	25	0	0
75	74	10	4	100	16	40
85	90	20	20	400	400	400
$\sum x = 717$	$\sum y = 721$	$\sum X = 2$	$\sum Y = -49$	$\sum X^2 = 1414$	$\sum Y^2 = 1865$	$\sum XY = 1393$

First we will calculate the standard deviation of 'x' and 'y' i.e. $\boldsymbol{\sigma_x}$ and $\boldsymbol{\sigma_y}$ and coefficient of correlation $'r'$.

Substituting the values from the Calculation Table in the formulas for coefficient of correlation and standard deviation, we obtain,

$$\sigma_x = \sqrt{\frac{\sum X^2}{n} - \left\{\frac{\sum X}{n}\right\}^2} = \sqrt{\frac{1414}{11} - \left\{\frac{2}{11}\right\}^2} = \mathbf{11.34}$$

$$\sigma_y = \sqrt{\frac{\sum Y^2}{n} - \left\{\frac{\sum Y}{n}\right\}^2} = \sqrt{\frac{1865}{11} - \left\{\frac{-49}{11}\right\}^2} = \mathbf{12.24}$$

$$r = \left[\frac{n\sum XY - \sum X . \sum Y}{\sqrt{\{n\sum X^2 - (\sum X)^2\}.\{n\sum Y^2 - (\sum Y)^2\}}}\right]$$

Thus,

$$r = \left[\frac{11 \times 1393 - (2) \times (-49)}{\sqrt{\{11 \times 1414 - (2)^2\}.\{11 \times 1865 - (-49)^2\}}}\right] = \mathbf{0.92}$$

Now as regression coefficient of x on y is b_{xy} and y on x is b_{yx} we have

$$b_{xy} = r\,\frac{\sigma_x}{\sigma_{xy}} = \frac{(0.92).\ (11.34)}{12.24} = 0.85$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x} = \frac{(0.92).\ (12.24)}{12.34} = 0.99$$

If the mean of x-series and y-series are $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}$, respectively, then

$$\bar{X} = 65 + \frac{\sum X}{n} = 65 + \frac{2}{11} = 65.2$$

$$\bar{Y} = 70 + \frac{\sum Y}{n} = 70 + \frac{(-49)}{11} = 65.55$$

$$\mathbf{y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})}$$

Hence, the equation of regression line of y on x is:

$$y - 65.55 = (0.99) . (x – 65.2)$$

$$y = (0.99) . x + 1.002$$

Also, the equation of regression line of y on x is:

$$\mathbf{x - \bar{x} = r \frac{\sigma_x}{\sigma_{xy}} (y - \bar{y})}$$

Therefore, the equation of regression line of x on y is:

$$x - 65.2 = (0.85) . (y – 65.55)$$

$$x = (0.85) . y + 9.48$$

Example 7: For certain X and Y series which are correlated, the two lines of regression are:

$$5X – 6Y + 90 = 0$$

$$15X – 8Y – 130 = 0$$

Find the means of the two series and the correlation coefficient.

Solution:

(i) Finding mean of the two series:

$$5X – 6Y = – 90 \quad \text{. . .(i)}$$

$$15X – 8Y = 130 \quad \text{. . .(ii)}$$

Multiplying equation (i) by 3, we get

$$15X – 18Y = -270$$

$$15X – 8Y = 130$$

$$– \quad + \quad –$$

$$–10Y = – 400$$

$$Y = 40$$

Putting the value of Y in eq. (i), we get X

$$5X – 6 (4) = -90$$

$$5X = –90 + 240$$

$$5X = 150$$

$$X = 30$$

(ii) Finding correlation coefficient:

Let us assume that equation (i) is the regression equation of X on Y;

$$5X = 6Y – 90$$

$$X = \frac{6}{5} Y – 18$$

$$b_{xy} = \frac{6}{5}$$

Taking eq. (ii) as the eq. of Y on X,

$$–Y = – 15X + 130$$

$$8Y = 15X – 130$$

$$Y = \frac{15}{8} X – \frac{130}{8}$$

$$b_{yx} = \frac{15}{8}$$

Since both the regression coefficients are exceeding one, our assumption is wrong. Hence, eq. (i) is the regression eq. of Y on X:

$$-6Y = -5X - 90$$

$$6Y = 5X + 90$$

$$Y = \frac{6}{5}X + 15$$

$$b_{yx} = \frac{5}{6}$$

Eq. (ii) is the regression eq. of X on Y;

$$15X = 130 + 8Y$$

$$X = \frac{130}{15} + \frac{8}{15}Y$$

$$b_{xy} = \frac{8}{15}$$

$$\mathbf{r} = \sqrt{\boldsymbol{b_{xy} X b_{yx}}}$$

$$= \sqrt{\frac{8}{15} X \frac{5}{6}}$$

$$= 0.667$$

Example 8: Height of fathers and sons are given below. Find the height of the son when the height of the father is 70 inches.

Height of Father (inches)	71	68	66	67	70	71	70	73	72	65	66
Height of Son (inches)	69	64	65	63	65	62	65	64	66	59	62

Solution:

To find the height of son when the height of father is given we have to fit the regression of son (y) on father (x).

Table for Calculation

Father (x)	**$X = x - \bar{x}$**	**X^2**	**Son (y)**	**$Y = y - \bar{y}$**	**Y^2**	**XY**
71	+2	4	69	+5	25	+10
68	–1	1	64	0	0	0
66	–3	9	65	+1	1	–3
67	–2	4	63	–1	1	+2
70	+1	1	65	+1	1	+1
71	+2	4	62	–2	4	–4
70	+1	1	65	+1	1	+1
73	+4	16	64	0	0	0
72	+3	9	66	+2	4	+6
65	–4	16	59	–5	25	+20
66	3	9	62	–2	4	+6
$\sum x = 759$	$\sum X = 0$	$\sum X^2 = 74$	$\sum y = 704$	$\sum Y = 0$	$\sum Y^2 = 66$	$\sum XY = 39$

$$\bar{x} = \frac{704}{11} = 64$$

$$\bar{y} = \frac{759}{11} = 69$$

$$r\frac{\sigma_y}{\sigma_x} = \frac{\sum XY}{\sum X^2} = \frac{39}{74} = 0.527$$

We know that the equation of regression line of y on x is given by:

$$\mathbf{y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

$$y - 64 = 0.527\,(X - 69)$$

$$y - 64 = 0.527\,X - 36.36$$

$$y = 0.527X + 27.64$$

For x= 70, y is given by:

$$y = 0.527\,(70) + 27.64$$

$$= 64.53$$

Hence, the height of son when the height of father is 70 inches shall be 64.53 inches.

Properties of Regression Coefficient

1. The Coefficient of Correlation is the Geometric Mean of the two coefficient of Regressions.
2. Both the coefficients must be of same sign i.e. either both are positive or both are negative, but one negative and other positive is not possible.
3. Arithmetic Mean of coefficient of regression is greater than the coefficient of correlation.
4. The coefficient of regression is independent of change of origin but not of scale.
5. The correlation coefficient and the two regression coefficients have same sign.
6. Both the coefficient of regressions cannot be greater than one and product cannot be greater than one i.e. $b_{xy} \cdot b_{yx} \leq 1$. But, one of them can be greater than one, individually; provided the above condition is satisfied.

3.6 Difference between Correlation and Regression

Following are the differences between correlation and regression.

- Correlation coefficient is a symmetrical function between x and y but the regression coefficient are not symmetrical functions between x and y.
- Correlation coefficient is independent of change of scale and origin, whereas, the regression coefficients are independent of the origin but not of scale.
- Correlation coefficient is a relative measure of relationship, whereas, regression coefficient is an absolute measure of relationship.
- Correlation coefficient is merely a tool of ascertaining the degree of relationship between two variables whereas in regression analysis one variable is taken as dependent while the other as independent; to estimate the value of one from another.

3.7 Summary

Correlation and Regression are important statistical tools for measuring the degree of association between series of pairs of observations of two or more variables. These tools are used for studying and measuring the extent of relationship between two or more variables.

Correlation is the possible relationship between two series of observations such that the changes in the values of one series are accompanied by changes in the values of other series. Two variables are said to be positively correlated if an increase (or decrease) in the value of one variable is accompanied by an increase (or decrease) in the value of other variable. Two variables are said to be negatively correlated if the increase (or decrease) in the value of one variable is accompanied by decrease (or increase) in the value of other variable. Coefficient of Correlation shows the strength (magnitude) and direction (direct or indirect) of the relationship between two variables. It stands for relative measure of relationship between two variables.

Regression is used to measure the average relationship between the correlated variables. It uses the known value of one variable to predict or estimate the unknown value of the other variable. Regression stands for absolute measure of relationship. In Linear Regression, only two variables are involved in a relationship. The Method of Least Squares was used to find the best regression lines, which gives best fit for the set of observations. By the help of regression lines, we are able to estimate the value of x by the value of y and vice versa.

3.8 Key Words

- **Bivariate Distribution** - a distribution in which each individual of the set can generate two values.
- **Scatter Diagram** - a diagram of plotted points that show relationship between two sets of data.
- **Linear Correlation**- Correlation between two variables in which relationship is in the form of polynomial of one degree.
- **Coefficient of Correlation** - measure the intensity of relationship between two variables.
- **Coefficient of Regression** - indicate relation of average of one variable with respect to the average of other variable (Assumed to be independent).

3.9 Self Assessment Test

1. Calculate the coefficient of correlation for the following observations:

X	5	7	8	4	9	3	2	5	4	3
Y	2	4	5	5	6	5	4	4	3	2

(Ans: 0.47)

2. Calculate the coefficient of correlation between the values of x and y from the following:

X	1	3	5	7	8	10
Y	8	12	15	17	18	20

(Ans: 0.98)

3. Calculate the coefficient of correlation from the following data:

X	100	200	300	400	500	600	700
Y	30	50	60	80	100	110	130

(Ans: 0.997)

4. Eight student obtained the following marks (out of 100) in Maths and Chemistry.

Marks in Maths	15	13	27	45	20	60	20	75
Marks in Chemistry	50	30	55	25	30	10	30	70

Calculate the rank correlation coefficient.

(Ans: 0.02)

5. The marks secured by recruits in the selection test (X) and in the proficiency test (Y) are given below:

X	10	15	12	17	13	16	24	14	22
Y	30	42	45	46	33	34	40	35	39

Calculate the rank correlation coefficient

(Ans: 0.4)

6. The coefficient of rank correlation of marks obtained by 10 students in statistics and accountancy was found to be 0.2. It was later discovered that the difference in ranks in two subjects obtained by one of the students was wrongly taken as 9 instead of 7. Find the correct coefficient of rank correlation.

(Ans: 0.394)

7. Heights of fathers and sons (in inches) are given in the following table:

Heights of father	65	66	67	67	68	69	71	73
Heights of Son	67	68	64	68	72	70	69	70

Obtain the two lines of regression and calculate the expected average height of the son when the height of father is 67.5 inches.

(Ans: $y = 0.421x + 39.29$; $x = 0.524y + 39.29$; height of son is 68.19)

8. For 50 students of a class the regression equation of marks in Statistics (X) on the marks in Accountancy (Y) is $3Y - 5X + 180 = 0$. The mean marks in accountancy is 44 and variance of marks in Statistics is $9/16^{th}$ of the variance of marks in Accountancy. Find the mean marks in Statistics and the coefficient of correlation between marks in two subjects.

(Ans: + 0.8)

9. The following table shows the ages (X) and blood pressure (Y) of 8 persons:

X	52	63	45	36	72	65	46	25
Y	62	53	51	25	79	43	60	33

Obtain the regression equation of Y on X and find the expected blood pressure of a person who is 49 years old.

(Ans: 49.502)

10. From the following data given below find:

(a) The two regression equations

(b) The coefficient of correlation between the marks in Economics and Statistics.

(c) The most likely marks in Statistics when the marks in Economics are 30.

Marks in Economics	25	28	35	32	31	36	29	38	34	32
Marks in Statistics	43	46	49	41	36	32	31	30	33	39

(Ans: (a) Regression eq. X on Y: X = 40.892 – 0.234Y; Regression eq. Y on X: Y = 59.248 – 0.664X; – 0.394; 39)

3.11 References

- ‘Introduction to Mathematical Statistics’, P.G. Hoel, John Wiley and Sons.
- ‘Methods of Correlation Analysis’, Ezekiel, John Wiley and Sons.
- ‘Mathematical Statistics’, O.P. Gupta, Kedarnath Ramnath and Co.

Unit - 4 : Business Forecasting and Time Series

Unit Structure:

4.0 Objectives

After studying this unit, you should be able to understand :

- The concept of business forecasting
- The objectives and importance of forecasting in business
- The concept of Time Series Analysis
- Utility of Time Series
- The various components and analysis of time series
- The measurement of trend

4.1 Introduction

Forecasting refers to the estimation of future based on the data related to past years. Forecasting may be done related to various areas of economy, business and other allied areas.

Forecasting is more important for business decision making. The success of business depends upon the prompt decision making based on appropriate data collection, in depth analysis and interpretation. Analysis is generally related to the past data and finding out trends for estimating future movements of demand, sales, cost and other concerned areas of business.

"Business forecasting refers to the statistical analysis of the past and current movements in a given time series, so as to obtain clues about the future pattern of the movements.**"**

—Neter and Wasserman

Following features may be considered from above explanation:

1. It is based on past data.

2. Business forecasting may be done for short period or long period.

3. Business forecasting may be related to particular variable or multiple variables.

4. Business forecasting can be useful in business analysis and decision making.

4.2 Objectives and Importance

Objectives of Forecasting in Business

Forecasting may become an important tool in business. In today's era, entrepreneurship is promoted by government and various subsidies are planned as well as provided for initiation. Thus, starting up a new venture or new business has became an easier task. But initiation alone is not enough, the real success of the business depends upon how well the entrepreneur uses his following skills for business forecasting:

- Conceptual Skills: It refers to the basic understanding of the operating activities of the business. It helps to identify the critical success factors for the business and also help to locate those variables which need proper attention and apt forecasting.
- Diagnostic Skills: It requires appropriate method of analyzing the data collected. If an entrepreneur lacks these skills then he may not find out the root cause of the problem and may not be able to take appropriate decision.

The above mentioned skills will promote approximate forecasting which would further decide the future success and failure of the business. Forecasting helps in minimizing the uncertainties related to future events. Forecasting also acts as a foundation for decision making related to – capital investment, price change, financing source, cost of capital, innovations in production techniques, markets, product design etc. It can be safely assumed that accuracy in forecasting cannot be always 100%. This is because the future is uncertain and uncertainty may be reduced but cannot be eliminated completely. Thus, a margin of error is allowed which is referred to as probable error.

Importance of Business Forecasting:

Business forecasting assumes great relevance when its results are implemented in various functions of management:

- **Planning**: The planning is based on various assumptions about future conditions. Forecasting helps in estimating the future conditions and thus helps in revising the planning premises. It provides stability of plans for a longer span of time.
- **Better Information System**: Forecasting is useful in gaining information about diverse fields which can be used in strategy formulation, planning marketing activities, formulating financial policies and improving the operating efficiency.
- **Improving productivity and profitability:** By analyzing the forthcoming events, productivity and profitability can be improved by eliminating waste such as:
- Plant shut-downs
- Lost sales
- Lost customers
- Expensive expediting

- Missed strategic opportunities.
- **Financial and Strategic Importance:** Forecasted data may be used in better strategy formulation as marketing, production, demand figures may be extrapolated. Financial policies can be formulated based on forecasted data which may result in better allocation of resources, increased profitability and higher return on investment.

4.3 Methods of Business Forecasting

Business forecasting may be done by various methods. As per the need of the business, appropriate method should be selected. Following important methods are discussed below:

Main Methods of Forecasting:

1. Time Series Analysis:

Time series analysis may be used for the purpose of business forecasting. Forecasting through this method can be done only when a series of data for several years is available. This method assumes that the business data reflects a trend which may be long term, cyclical or seasonal in nature. This trend may be identified by using this method and then future trend is predicted on its basis. Time series analysis is based upon additive or multiplicative model. (Discussed in detail in the later half of this unit)

Merits

i. It is an easy method to apply if appropriate data series is available.

ii. As the future is forecasted on the basis of past business data, the forecasting will be more reliable.

Demerits

i. In this method, the forecasting is based on guess work not on a scientific method because the past and present conditions are rarely found to be similar.

ii. It is very difficult to select the past period with the same business conditions like present.

2. Survey Method:

Survey means to approach the respondents comprehensively or selectively. Survey may be of two types: Census survey and Sample survey. Census survey means to approach each person under the study to collect the required information. It consumes more time and money. On the other hand, sample survey may be conducted in which a sample of few persons is selected from the population under study randomly or according to the judgment of the researcher. Sample survey is convenient for data collection and consumes less time and money.

Field surveys can be conducted to gather information about the intentions and attitudes of the concerned people for the future. Such surveys may be conducted amongst consumers about the likely expenditures on, say consumer durables. Both, quantitative and qualitative information may be gathered.

The survey may be conducted by using questionnaires, interviews or schedules. The data collected is then tabulated for analysis. On the basis of analysis future trend may be predicted.

3. Opinion Poll:

This method is used when valuable information is assumed to be obtained not from the customer but from others who are related closely to the subject under study. This method is useful in behavioral or marketing data collection for future forecasting.

Opinion poll is the survey of opinion of knowledgeable persons or experts in the field whose views carry lot of weightage in the area of study. For example, a survey of opinion of the sales representatives, wholesalers or marketing experts may be helpful in gathering data about the consumers intention or expectations and then on its basis demand projections may be made.

This approach is the vicarious approach i.e. using others experience to gather knowledge. This method is reliable as the forecasting based on this method has a greater tendency of accuracy.

4. Exponential Smoothing

This method reduces the extremeness of past values and gives more importance to the current values. Exponential smoothing is special kind of weighted averages which is more useful in short-term forecasting. This method gives less weights to the past values and higher weights to the current values. The effect of past data is automatically reduced as more weightage is given to recent year values. This method is more useful and gives reliable results on which effective business decisions may be based. Thus, this method is regarded as the best method of business forecasting as compared to other methods.

Merits

i. Forecasting is made on the basis of past conditions so it is more reliable.

ii. This method is helpful in short- term forecasting.

iii. This method may be conveniently applied on computer.

Demerits

i. It cannot be successfully applied for long term forecasting.

ii. Present conditions are given more weight-age than the past conditions.

iii. This method involves heavy calculations as it involves calculation of moving average and then applying weights for further calculations.

5. Regression Analysis:

It is a statistical technique used to find the nature of relationship between variables. The identification of relationship can be used for forecasting of either of the variables. There may be two or more than two variables under study. For more than two variables, multiple regression is used.

Under this method, first the variables are identified as dependent and independent variables. Then, the value of dependent variable may be forecasted by identifying its nature of relationship with the independent variable. The relationship is expressed in two forms of equations stated as follows:

i) X on Y : The value of dependent variable 'X' may be worked out when independent variable 'Y' is known.

ii) Y on X : The value of dependent variable 'Y' (assuming it as dependent) may be worked out when values of variable 'X' are known and assumed as independent.

6. Barometric Techniques:

Barometer means which indicates about something. A barometric technique involves the usage of economic barometers or business barometers for forecasting purpose. Business barometers are those factors which represent the change in the economic conditions of the economy and thus affect the individual business unit as a whole. A study of these barometers can be used in predicting the future trends in business conditions.

Some of the economic indicators are – Gross Domestic Product, National Income, Consumer Price Index etc.

The business barometers may be classified into three categories:

i) General Index of Business Activity: These barometers are related to general business activities. A separate index for every type of unique business activity is constructed which help in the formulation of the economic policies of the country.

ii) Barometers for Specific Business Activities: These indices are worked for specific infrastructural business activities which act as a supplementary index to the above indices. E.g. Iron & steel, petrochemical business etc.

iii) Barometers for individual business firm: These indices are calculated in relation to specific business firm and may be used in estimating future variations in the business activities.

7. Modern Econometric Method:

This method is purely mathematical and statistical based. It involves many sets of simultaneous equations to be prepared. Solving these equations manually is very difficult, so the usage of advanced technology is a must. Computer softwares and advanced computing programmes are now a day very popularly used; using this method for business forecasting became easy.

Merits:

i. Accurate and reliable results are obtained under this method because it is a scientific method where computer is used for calculations.

ii. This method is more logical as it explains the interrelationship between various economic variables.

Demerits:

i. This method is difficult and complicated as it involves a number of equations to be formulated and then solving them.

ii. If appropriate data series is not available, it will make this method unusable.

iii. It is based on growth model which is difficult to build up.

Activity A

1. Identify the problems pertaining to selection of the method for forecasting sales in any organization familiar to you and choose the most appropriate method.

4.4 Time Series

"A time series may be defined as a sequence of repeated measurements of a variable made periodically through time." **—Cecil H. Mayers**

"A time series may be defined as a sequence of values of some variable corresponding to successive points in time." **—W.Z.Hersch**

Time series is a collection of readings belonging to different time periods in the order of their occurrences.

Following are the examples of time series:

1. Monthly consumption.

2. Annual Production.

The following is the time series data which can be further used for forecasting:

Year	Sales (in lakh)
2006	10
2007	12
2008	16
2009	8
2010	6
2011	14

4.5 Importance and Utility

Importance or Utility of Time Series Analysis:

- **Based on analysis of past behavior**: - It is assumed that the future trends are based on the historical trends. Thus, such analysis will be extremely helpful in predicting the future behavior.

- **Creating a road map for the future**: It helps in planning future operations. Business operations can be run more efficiently and effectively by predicting the future.

- **It facilitates comparison**: - Different time series are often compared and important conclusions drawn there from. The trends reflected may be different like one time series may have seasonal variations and the other time series may have more of irregular fluctuations component.

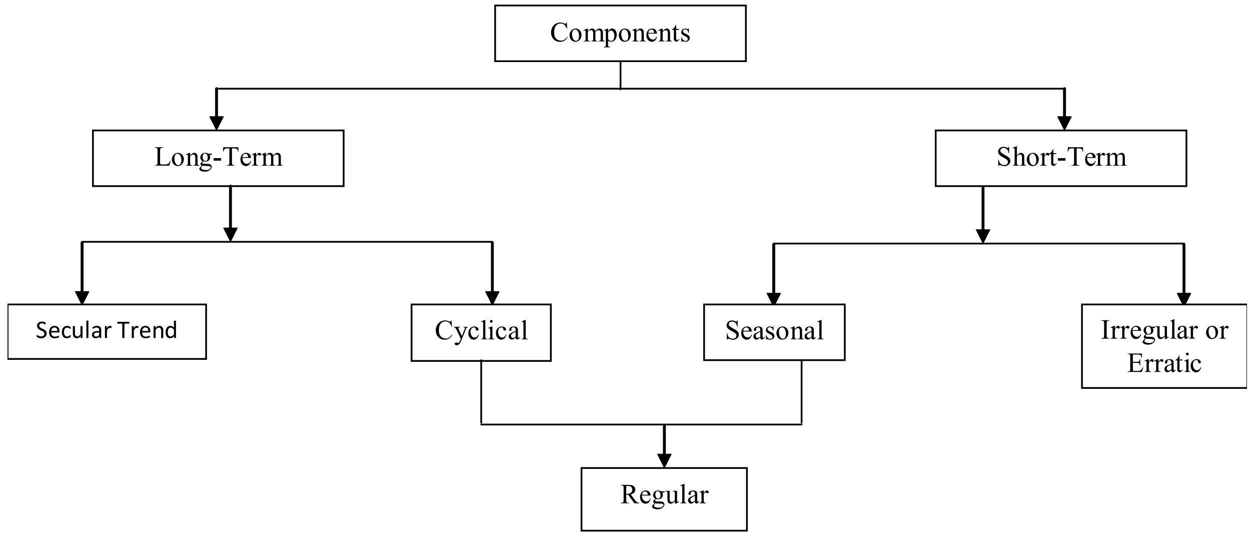
- **It helps in planning future operations:** - Plans for the future can not be made based on the result of time series. Statistical techniques have been evolved which enable time series to be analysed in such a way that the influences which have determined the form of that series may be ascertained. If the regularity of occurrence of any feature over a sufficient long period could be clearly established then, within limits, prediction of probable future variations would become possible.

- **It helps in evaluating current actions:** - The actual performance can be compared with the expected performance and the cause of variation analysed. For example, if expected sale for 2010-11 was 1,000 laptops and the actual sale was only 700, one can investigate the cause for the shortfall. Time series analysis helps to analyse one variable at a time as it assumes other things constant. For example, if we know how much is the effect of seasonality on business we may find ways and means of ruling out the seasonal influence or decreasing it by producing commodities with complementary seasons.

4.6 Components of Time Series

Time series analyses many variations in the available data. The variations may be of different forms. These diverse changes can be categorized under following four categories, referred to as 'Components of Time Series'.

1. Secular Trend
2. Cyclical Trend
3. Seasonal Variations
4. Irregular Variations



The components of time series as depicted by the above diagram represent fluctuations over time in various business activities which may be long term or short term.

- **Long Term**: The fluctuations which are long term in nature may be worked out in two ways- Secular trend and Cyclical trend.
- **Short Term**: The fluctuations of short term nature may be represented seasonally or in irregular formats.

1. Secular Trend: -

General movement persisting over a long period of time (several time periods) is called secular trend. Secular trend movements are due to the factors such as population change, technological progress and large scale shifts in consumer tastes.

Features:

- It involves long term steady trends created by 'Growth factors' or 'Decline factors'.
- They represent slow and continuous trends.
- The short term happenings do not affect this trend.
- Such trends may be notified widely in the economic conditions of the country.

2. Cyclical Movements or Cyclical Variations: -

Cyclical movements refer to the long term (over several time periods / years) movements which are consistent and generally periodic in nature.The values increase over some years reach peak and start downward movement and touch a bottom; again start increasing and reach a top; this complets one business cycle. The time period of business cycles varies and so does the top and bottom levels. These movements are important in analysis of business activities as business cycle represents cyclical movements of ups and downs.

Features:

- They depict the turning points of the business activities which further help in systematic future planning.
- Normal range of these variations is from 2 to 15 years.
- It analyzes the expansion and contraction phase of the economic activities.

3. Seasonal Movements or Seasonal Variations: -

These refer to the identical patterns that a time series appears to follow during corresponding months or quarters of successive years. The type of fluctuation, which completes the whole sequence of changes within the span of a year and has the same pattern year after year, is called seasonal variations.

Features:

- The seasonal movements are short term in nature and are repeated perpetually every year.
- The major factors causing these fluctuations are:
  - a) Climate and weather
  - b) Customs and traditions
  - c) Habits
  - d) Fashion
- An analysis of these variations will help to tap the unused resources and their optimum utilization.

4. Irregular or Random Movements: -

The variations which do not follow a defined pattern and their reoccurrence is not certain, nor may be depicted is termed as 'Irregular Variations'. These fluctuations occur generally due to contingent happenings or unexpected situations.

Features:

- They cannot be predicted as they do not repeat in a definite pattern.
- They are caused by earthquakes, floods etc. and very rapid technological changes.
- These variations may act as a turning point of the business which may be taken as a threat or an opportunity.

Time Series Models

An analysis of time series may lead to a combination of the above four variations and so each variation must be identified to work out a trend value for referring future values.

There are two types of time series models:

1. Additive Model : It assumes that the original data is a sum of the four variations.

$Y = T + S + C + I$

Y = Original Data

T = Trend value

S = Seasonal Variations

C = Cyclical Variations

I = Irregular Variations

2. Multiplicative Model or Classical Model: It assumes that the original data is a product of the four components. This model is widely used.

$$Y = T \times S \times C \times I$$
$$= TSCI$$

4.7 Preliminary Adjustments for Time Series

Preliminary adjustments refers to those changes required to eliminate the effect of abnormal events or certain peculiarities which may effect the time series data. These adjustments are explained below:

1. Calendar Adjustments: - To eliminate certain differences which are caused by peculiarities of our calendar like – no. of days in month, holidays, festivals etc.

Most of the time series are available in a monthly form and it is necessary to recognize that the month is a variable time unit. The actual length of the shortest month is about 10% less than that of the longest, and if we take into account holidays and weekends, the variation may be even greater. When we study the production level or sales on a monthly basis, identifiable variations may be observed in different months.

Thus the purpose of adjusting for calendar variation is to eliminate certain counterfeit differences which are caused by peculiarities of our calendar. The adjustment for calendar variation is made by dividing each monthly total by the number of days in the month.

Daily average for each month = Monthly Total / No. of working days

Comparable (adjusted) monthly data may then be obtained by multiplying each of these values by 30.4167, the average number of days in a month. (In a leap year this factor is 30.5)

2. Population Change Adjustments: -

Certain types of data call for adjustment for population changes. Changes in the size of population can easily distort comparisons of income, production and consumption figures. For example, national income may be increasing year after year, but per capita income may be declining because of greater pressure of population.

In such cases where it is necessary to adjust data for population changes, a very simple procedure is followed, i.e., the data are expressed on a per capita basis by dividing the original figures by the appropriate population totals.

3. Adjustment for price changes: - Adjustment for price rise and fall should be made.

An adjustment for price changes is necessary whenever we have a value series and are interested in quality changes alone. Because of rising prices the total sales proceeds may go up even when there is a fall in the number of units sold. For example, if in 2010, 1,000 units of a commodity that is priced Rs. 20 are sold, the total sale proceeds would be 1,000 x 20 = Rs. 20,000. Now suppose in 2011 the price of the commodity increases from Rs. 20 to Rs. 22. If the sales do not decline, the total sales proceeds will be 1,000 x 22 = Rs. 22,000. This increase in sales proceeds, i.e., Rs. 2,000, is not due to increase in the demand but due to the rise in price from Rs. 20 to Rs. 22.

The effect of price changes can be eliminated by dividing each item in a value series by an approximate price index.

4. Comparability: - For any meaningful analysis of time series, it is necessary to see that the data are strictly comparable throughout the time period under investigation.

For example, if we are observing a phenomenon over the last 20 years, the comparability may be observed by differences in definition, differences in geographical coverage, differences in the method adopted, change in the method of reporting etc. Like, if we compare the profit of a concern in March 2000, and then later on after a few years noticed the profit, there may be a change in the method of charging depreciation, treatment of revenue items as capital items etc. Such changes would make the comparison meaningless.

Activity B

Take a time series data for 10 years pertaining to production or employee turnover in any organization familiar to you and point out the preliminary adjustments required.

4.8 Methods of Finding Trends

There are various methods for finding trends which are given below:-

Measurement of Secular Trend/ Methods of Finding Trend:

Various methods that can be used for determining trend are:

1) Freehand or Graphic Method

2) Statistical (Semi Average Method)

3) Algebraic (Least Square) Method

4) Moving Average Method

1) Free hand curve method: - Plot points on the graph which represents the trend in the time series. Draw smooth and free hand line among them.

Merits

- This method is very simple to apply as it does not involve any calculations.
- It is flexible as no rigid boundaries are framed for its application. This method may be applied even if some data is missing as approximate curve may be obtained.
- It saves the time of the researcher as trend may be analyzed without making any cumbersome calculations.

Demerits

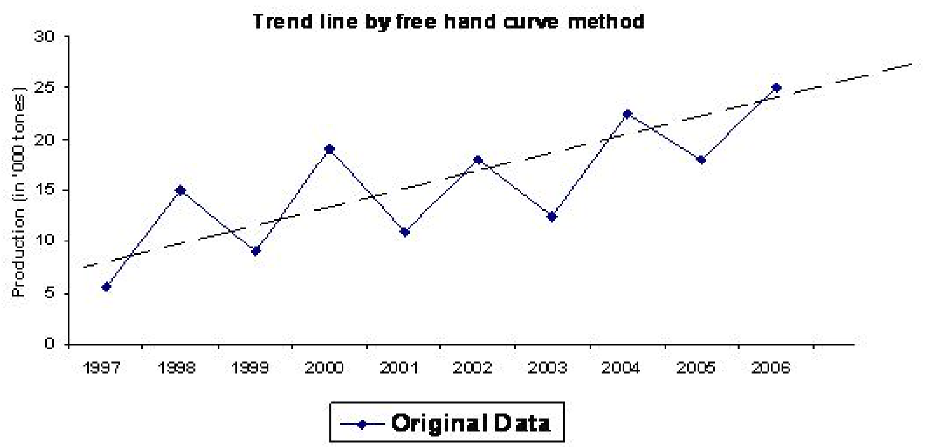
- The usage of this method leads to lack of exactness in the estimated trend.
- This method is highly subjective as the trend line obtained depends on the personal judgment of the researcher.

Illustration: From the figures given below, fit a trend line by free hand curve method:

Year	Production ('000 tones)	Year	Production ('000 tones)
1997	6	2002	19
1998	15	2003	14
1999	10	2004	23
2000	18	2005	18
2001	12	2006	25

Solution:

Trend Line



2) Method of Semi Averages: - Under this method the time series data is divided in two equal parts. The values of both parts are added and their mean is worked out. Plot each on the middle year of each part and obtain trend line and extend it both sides. If number of years are odd then middle year is left and remaining values are divided in 2 parts.

Merits

- This method is simple to apply and involves less calculations as compared to other methods.
- It is objective method as same results are obtained even if it is applied by different researchers.

Demerits

- Limitations of arithmetic average apply.
- If odd number of years is there, then results are not accurate as the middle year is ignored.
- It assumes straight line relationship between plotted points which may be unrealistic in some cases.

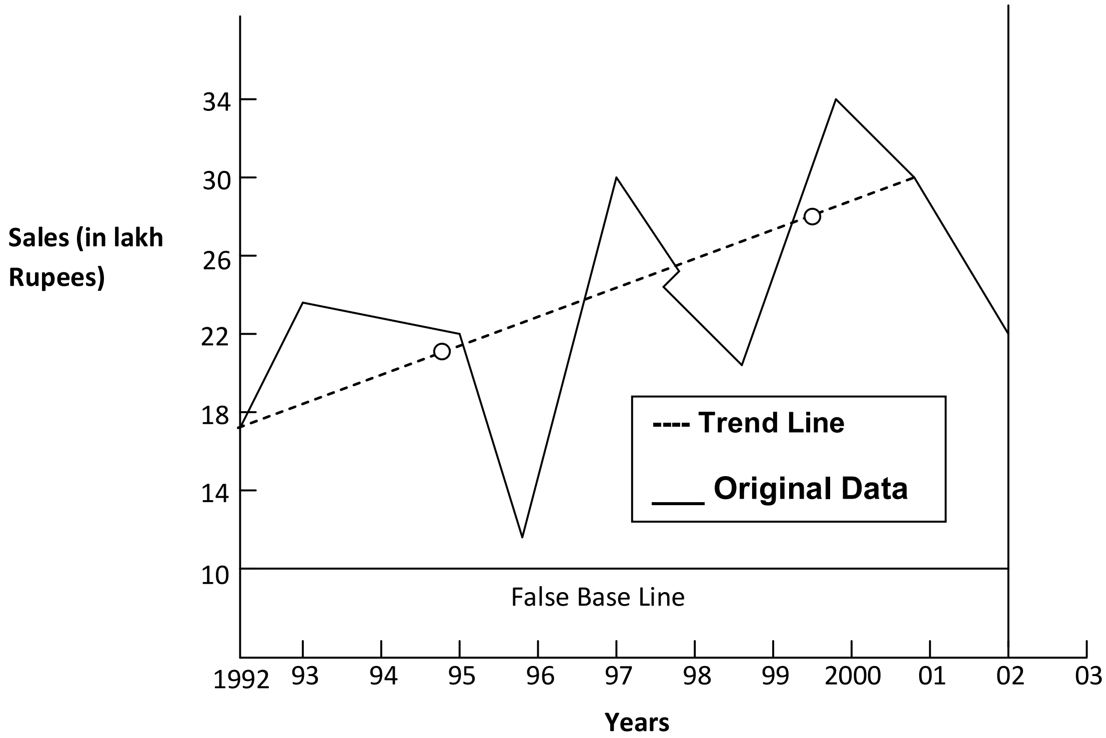
Illustration: Fit a trend by the method of semi-averages to the following data:

Year	Sales (Rs. in lakh)	Year	Sales (Rs. in lakh)
1993	15	1999	28
1994	22	2000	20
1995	20	2001	34
1996	11	2002	30
1997	27	2003	23
1998	31		

Solution: In the above problem, 11 years are given as such we will leave the 6th year and the remaining 5-5 years will be taken for calculating the averages.

Calculation of Semi-Average

Year	Sales (Rs. in lakh)	Total	Semi-Average	Median Year
1993	15			
1994	22			
1995	20	95	95/5 = 19	1995
1996	11			
1997	27			
1998	**31**			
1999	28			
2000	20			
2001	34	135	135/5 = 27	2001
2002	30			
2003	23			



3) Method of least Squares or fitting a straight line trend: - The line obtained by the method of least squares is called "Line of best fit". The sum of the squares of the deviations of the actual and computed values is least from this line.

Conditions:

Apply Formula [2 eq.], get a & b and put in $Y_c = a + bx$ and get trend values.

After eliminating trend, we are left with Cyclical & irregular variations.

Monthly increase = b/12

Illustration: The following table represents profits of a multinational company. Fit a straight line trend and estimate the figure of profit of the year 2004:

Years	:	1997	1998	1999	2000	2001	2002	2003
Profit (000' Rs.)	:	300	700	600	800	900	700	1000

Solution:

- $\Sigma(Y - Y_c)^2$ is least

Fitting a Straight Line Trend: -

$Y_c = a + bX$

Y = Trend value to be computed
X = Unit of Time (Independent Variable)
b & a = Constant to be calculated.

To find 'a' and 'b' the following two normal equations are used:

$\Sigma Y = Na + b\Sigma X$ .... (i)

$\Sigma XY = a\Sigma X + b\Sigma X^2$ .... (ii)

Year	Y	X	XY	X^2	Y_C	$\frac{??}{???}$ x 100
1980	80	0	0	0	84	95.23
1981	90	1	90	1	86	104.65
1982	92	2	184	4	88	104.54
1983	83	3	249	9	90	92.22
1984	94	4	376	16	92	102.17
1985	99	5	495	25	94	105.31
1986	92	6	552	36	96	95.83
N = 7	ΣY = 630	ΣX = 21	ΣXY = 1946	$ΣX^2 = 91$		

Apply formula, get a & b and put in $Y_c = a + bx$ and get trend values.

After eliminating trend, we are left with seasonal cyclical & irregular variations.

First calculate value of b

$$b = \frac{NΣxy - ΣxΣy}{NΣx^2 - (Σx)^2}$$

$$= \frac{7 \times 1946 - 21 \times 630}{7 \times 91 - (21)^2}$$

$$= \frac{13622 - 13230}{637 - 441}$$

$$= \frac{392}{196}$$

b = 2

Substituting the values in (1) above

$$a = \frac{630 - 2 \times 21}{7}$$

= 84

Therefore, the trend equation is

$Y_c = 84 + 2x$

Therefore the trend value for the year 1980; substituting x = 0 in example.

= 84

Similarly trend values can be calculate for other years; and the results are shown in the column under Y_c.

Note: If annual change in trend values in given by b then monthly change by $\frac{b}{12}$ or quarterly change in given by $\frac{b}{4}$

Similarly, if the monthly change in trend value in b then the yearly change is given by 12b And quarterly change is given by 4b.

Eliminating the effect of trend, the detrended values (series) are calculated using. $\frac{Y}{Y_c}$ x 100 for each year. The results are shown in last column.

Here we write X = 1, corresponding to first year, 1997, X = 2 for the second year 1998 and so on. The values of profit are denoted by Y.

Merits and demerits of method of least squares:

Merits: 1) There in no possibility for subjective ness as all calculations are formula based and personal biasness of the researcher cannot affect the results.

2) The various trend values can be obtained for all time periods covered under the study.

3) Line of best fit is obtained. This line gives more appropriate estimations about the future trend values.

Demerits: 1) There may be different approaches to this method. Thus it becomes difficult to select.

2) This method involves heavy calculations and so it is quite time consuming.

3) The predictions about future economic values are based only on trend.

4) This method is not flexible as the necessary conditions are pre defined with no scope for any alterations..

4) Method of Moving Averages: -

Under this method, the average value for a particular number of years (3, 4, 5 or more) is calculated and it is considered to be the trend value for the unit of time falling at the middle of the period covered.

This is the most common method used to measure secular trend. While using this method it is important to select period for moving averages; For Example – 3 yearly moving averages, 5 yearly moving averages, 8 yearly moving averages etc. This period can be even or odd. It is calculated as follows:

Firstly we decide average period. Example: 3 years moving average

If period is in odd like 3 years then take the average of the first three values and place this against the middle year, i.e. 2nd year.

After this, find the average of values leaving the values of the first year and including the value of the fourth year and place it against the 3rd year. Continue this process till you reach the last value of the series.

These moving average figures are also called trend value.

When the period of moving average is even, i.e. 4, 6, 8 years, the following steps are adopted:

If 4 yearly period, add the first four values and place this against in the middle of the second & third year.

After this, find the sum of values leaving the value of the first year and including the value of the fifth year and place it against in the middle of the third and fourth, complete this process till the end.

Now add the first and the second four yearly totals place it against the 3rd year. Similarly continue this process till end.

Divide the sums derived thus by twice the period of moving average (8 in this case) respectively to get moving averages.

Illustration

Fit a trend line from the following data by 3 yearly moving average methods.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009
Production ('000 quintals)	9	11	10	12	14	8	9	10	9

Solution:

Trend Values by 3 yearly moving averages

Year	**Production in**	**3 Yearly moving totals**	**3 yearly moving averages**
2001	9	-	-
2002	11	30	10
2003	10	33	11
2004	12	36	12
2005	14	31	11.33
2006	8	27	10.33
2007	9	28	9
2008	10	-	9.33
2009	9		-

Here, we have to calculate three yearly moving averages. For this we first calculate the sum of first three values 9+11+10=30 and place it against the middle year i.e., the second year 2002. Then leaving out the first value add the values 11+10+12=33 and place it against the third year i.e., 2003. Thus every time we leave the first of the three and add the next one. These are called moving totals.

These moving totals are divided by 3 to get the moving averages. Moving average represent the trend values for the specified years.

Illustration

Find 4 yearly moving averages of the sales figure given below:

Year	2003	2004	2005	2006	2007	2008	2009	2000	2011
Sales (in '000 Rs.)	90	106	104	100	105	102	103	102	110

Solution:

Calculation of 4 yearly moving averages

Year i)	Sales (ii) (in '000 Rs.)	4 yearly Totales (iii)	4 yearly centered total (iv)	4 yearly centered average (iv/8)
2003	98	-	-	-
2004	106	408	-	-
2005	104	415	823	102.8
2006	100	411	826	103.2
2007	105	410	8221	102.6
2008	102	412	822	102.7
2009	103	417	829	103.6
2010	102	-	-	-
2011	110	-	-	-

Here, we have to calculate four yearly moving averages. For this we find 4 yearly moving totals and place them at the center. These four yearly moving totals are centered by taking two yearly moving totals i.e., the four yearly totals are added two at a time. First we add 408 + 415 = 823 and place it between 408 and 415 i.e., against the year 2005. Next add 415+411=826 and place it again 2006 and so on. To get the average we divide these centered totals by 8. These are called 4 yearly centered averages and represent the trend Value.

Merits and demerits of Moving Average:

Merits:

- The method is simple to understand and easy to adopt, as compared with the least square method, as it does require simple calculation..
- The moving average method is quite flexible in the sense that a few more observations may be added to a given data without affecting the trend values already obtained.
- The moving average has the advantages that it follows the several movements of the data and that its shape is determined by the data rather than the statistician's choice of a mathematical function.
- The moving average is particularly effective if the trend of a series is very irregular.

Demerits:

- Since the moving average is not represented by a mathematical function, this method cannot be used in forecasting which is one of the main objectives of trend analysis.
- Great care has to be exercised in selecting the period of moving average.
- They are useful only when the trend is linear or approximately so.

Thus, analysis of time series assumes great importance in business decision making. The increasing relevance of business forecasting has led to the development of such techniques. In developed countries, there are many forecasting agencies which provide professional assistance on these techniques. But in case of developing countries like India, there are hardly any forecasting agencies emerging. Although, there is a need of such agencies as business opportunities are growing progressively in India.

Methods for calculation of seasonal Variations

Most of the phenomena in economics and business show seasonal patterns. When data are expressed annually there is no seasonal variation. However, monthly data frequently exhibit strong seasonal movements and considerable interest attaches to devising a pattern of average seasonal variation. There are many techniques available for computing an index of seasonal variation, many of the simpler methods were devised prior to the development of electronic computers and were designed to sacrifice precision to the development of electronic computers and were designed to sacrifice precision for ease of computation. Any acceptable modern method for computing such an index probably will be programmed for a computer solution. The method should be designed to meet the following criteria.

1. It should measure only the seasonal forces in the data. It should not be influenced by the forces of the forces of trend or cycle that may be present.
2. It should modify the erratic fluctuation in the data with an acceptable system of averaging.
3. It should recognize slowly changing seasonal patterns that may be present and modify the index to keep up with those changes.

The following are some of the methods more popularly used for measuring seasonal variations:

1. Method of Simple Average (Weekly, Monthly or Quarterly)
2. Ratio- to- Trend Method
3. Ratio- to – Moving Average Method
4. Link Relative Method

1 **Method of Simple Average:-** Method of simple average is the simplest method of obtaining a seasonal index. Here are the following steps which are necessary for calculating the index.

1. Arrange the unadjusted data by years and months.
2. Find the totals of January , February, etc.
3. Divide each total by the number of years for which data are given. For example:- if we are given monthly data for five years then, we shall first obtain total for each month for five years and divide each total by 5 to obtain an average.
4. Taking the average of monthly averages as 100, compute the percentages of various monthly average as follows:

5. $$\text{Seasonal Index for January} = \frac{\text{Monthly average for January}}{\text{Average of Monthly averages}} \times 100$$

For example:-

Consumption ofmonthly electric power in million of kwh for street lighting in a big city during 2003-2007 is given below:

Year	Jan	Feb	March	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
2006	159	140	139	125	115	108	111	122	134	151	162	173
2007	171	154	150	134	124	118	121	131	144	160	171	183
2008	183	164	160	143	134	125	129	142	154	172	183	197

Find out seasonal variation by the method of monthly averages.

Solution:

CONSTRUCTION OF SEASONAL INDICES BY

THE METHOD OF MONTHLY AVERAGES

Consumption of monthly electric power

Months.	2006	2007	2008	Monthly totals For 3 years	Three yearly average	Seasonal variation
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Jan	159	171	183	513	171	116.4
Feb	140	154	164	458	152.7	104
March	139	149	160	448	149.3	101.7
April	125	134	143	402	134	91.8
May	115	124	134	373	124.3	84.7
June	108	118	125	351	117	79.7
July	111	121	129	361	120.3	81.9
Aug	122	131	143	395	131.7	89.7
Sep	134	144	154	432	144	98.1
Oct	151	160	172	483	161	109.6
Nov	162	171	183	516	172	117.1
Dec	173	182	197	552	184	125.3
			Total	5284	1761.5	1200
			Average	440.3	146.8	100

The above calculations are explained below:

1. Column No.5 gives the total for each month for 3 years.
2. In column No.6 each total of Column No.5 has been divided by 5 to obtain an average for each month.
3. The average of monthly averages is obtained by dividing the total of monthly averages by 12.
4. In column No.7, each monthly average has been expressed as a percentage of the average of monthly averages. Thus, the percentage for January

$$= 171/146.8*100 = 116.14$$

If, instead of monthly data, we are given weekly or quarterly data, we shall compute weekly or quarterly averages by following the same procedure as explained above.

Merits and Demerits of the Method of Monthly Average:

Merits:-

1. Simple average method is very suitable when materials are received in uniform lot quantities.
2. Simple average method is very easy to operate.
3. Simple average method reduces clerical work.

Demerits:-

1. If the quantity in each lot varies widely, the average price will lead to erroneous costs.

2. Costs are not fully recovered.
3. Closing stock is not valued at the current assets.

2 **Ratio- to- Trend Method: -** This method is used when cyclical variations are absent from the data, i.e. the time series variable Y consists of trend, seasonal and random components. Using symbols, we can write

$$Y = T.\ S\ \ .R$$

Various steps in the computation of seasonal indices are:

1. Obtain the trend values for each month or quarter, etc, by the method of least squares.

2. Divide the original values by the corresponding trend values. This would eliminate trend values from the data.

3.To get figures in percentages, the quotients aremultiplied by 100. Thus, we have three equations:

$$Y/\ T \times 100 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$T.\ S.\ R\ /\ T \times 100 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$S.\ R \times 100 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Q. Find seasonal variations by the ratio-to-trend method from the data given below:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
2006	15	20	18	17
2007	17	26	25	22
2008	20	29	27	24
2009	27	38	34	36
2010	40	46	43	41

Solution: For determining seasonal variation by ratio-to-trend method, first we will determine the trend for yearly data and then convert it to quarterly data.

CALCULATING TREND BY METHOD OF LEAST SQUARES

Year	Yearly Totals	Yearly average	Deviations from mid-year	*XY*	X^2	Trend Values
		Y	*X*			
2006	70	17.5	-2	-35	4	16
2007	90	22.5	-1	-22.5	1	22
2008	100	25	0	0	0	28
2009	130	32.5	1	32.5	1	34
2010	170	42.5	2	85	4	40

N=5 “$\sum Y = 140$ $\sum XY = 60$ $\sum X^2 = 10$

The equation of the straight line trend is Y = a+bY

a = $\sum Y/N = 140/5 = 56$

$b = \sum XY / \sum X^2 = 60/10 = 12$

Quarterly increment= 12/4=3

Merits and Demerits

It is an objective method of measuring seasonal variations. However, it is very complicated and doesn’t work if cyclical variations are present.

3 **Ratio- to – Moving Average Method: -** The ratio to moving average is the most commonly used method of measuring seasonal variations. This method assumes the presence of all the four components of a time series. Various steps in the computation of seasonal indices are as follows:

1. Compute the moving averages with period equal to the period of seasonal variations. This would eliminate the seasonal components and minimize the effect of random component. The resulting moving averages would consist of trend, cyclical and random components.
2. The original values, for each quarter (or month)are divided by the respective moving average figures and the ratio is expressed as a percentage, i.e. SR” = Y / M. A = TCSR / TCR’, where R’ and R” denote the changed random components.
3. Finally, the random component R” is eliminated by the method of simple averages.

Example 11.3.1 : Calculate the three and five year moving averages of the following data:

Year	**1970**	**1971**	**1972**	**1973**	**1974**	**1975**	**1976**	**1977**	**1978**	**1979**	**1980**	**1981**
Production	**18**	**19**	**20**	*22*	**20**	**19**	**22**	**24**	**25**	24	**25**	26

(**‘000** tons)

Year	Prediction y	3 yrs Moving Trend	3 yrs Moving Average	5 yrs Moving Trend	5 yrs Moving Average
1970	18	-	-	-	-
1971	19	57	19.0	-	-
1972	20	61	20.3	99	19.8
1973	22	62	20.6	100	20.0
1974	20	61	20.3	103	20.6
1975	19	61	20.3	107	21.4
1976	22	65	21.6	110	22.0
1977	24	71	23.6	114	22.8
1978	25	73	24.3	120	24.0
1979	24	74	24.6	125	25.0
1980	25	75	25.0	-	-
1981	26	-	-	-	-

Steps of Calculation

1) In Table 1 1.3.1 the figures in col. 3 are obtained as the sum of three consecutive values of col. 2. Thus the first moving total (M.T.) is 57 = 18 + 19 + 20 and is placed against 1971. The second moving total 61 = 19 + 20 + 22 is placed against 1972.

2) The three-year moving average(M.A.) in col. 4 is obtained by dividing the corresponding three-year moving total in col. 3 by 3, the period of the moving average. Thus 57 $\div$ 3= 19, 61 $\div$ 3 = 20.3, etc.

3) The five-year moving totals in col. 5 are obtained as the sum of five consecutive values in col. **2.** Thus the first moving total against the year 1972 is 99 = 18 + 19 + 20 + 22 + 20.

4) The five-year moving average in col. 6 is obtained by dividing the corresponding five-year moving total in col. 5 by 5,. Thus the moving average for 1975 is 107 $\div$ 5 = 21.4.

Merits and Demerits

This method assumes that all the four components of a time series are present and, therefore, widely used for measuring seasonal variations. However, the seasonal variations are not completely eliminated if the cycles of these variations are not of regular nature. Further, some information is always lost at ends of the time series

4. **Link Relative Method:** This method is based on the assumption that the trend is linear and cyclical variations are of uniform pattern. The link relatives are percentages of the current period (quarter or month) as compared with the previous period. With the computations of the link relatives and their average, the effect of cyclical and the random components is minimized. Further, the trend gets eliminated in the process of adjustment of chain relatives.

The following steps are involved in the computation of seasonal indices by this method:

1. Compute the Link Relative (L.R.) of each period by dividing the figure of that period with the figure of previous period. For example, Link relative of 3rd quarter=figure of 3 rd quarter /figure of 2nd quarter ×100.
2. Obtain the average of link relatives of a given quarter (or month) of various years. A.M. or M d can be used for this purpose. Theoretically, the later is preferable because the former gives undue importance to extreme items.
3. These averages are converted into chained relatives by assuming the chained relative of the first quarter (or month) equal to 100. The chained relative (C.R.)for the current period (quarter or month)= C.R. of the previous period ×L.R. of the current period / 100.
4. Compute the C.R. of the first quarter (or month) on the basis of the last quarter (or month). This is given by C .R. of the last quarter (month) × average L.R. of the first quarter (month) / 100

   This value, in general, is different from 100 due to long term trend in the data. The chained relatives, obtained above, are to be adjusted for the effect of this trend. The adjustment is to be done follows:

   a) Calculate
   Chain Relative (C.R.) of 1st month on the basis of last month

$$= \frac{CR\ of\ last\ month \times Average\ of\ link\ \mathrm{Re}lative\ of\ 1^{st}\ month}{100}$$

b) Find this is k
Difference between the two chain relatives
= 100- k [this can be +ve or -ve]

c) Make adjustment

Correction factor for each quarter = $\frac{100-k}{4} = CF$

Corrected Values:

Corrected C.R. of 1st quarter = 100

Corrected C.R. of 2nd quarter = CR of 2nd quarter $\pm$ 1X CF

Corrected C.R. of 3rd quarter = CR of 3rd quarter 2X CF

Corrected C.R. of 4th quarter = CR of 4th quarter 3X CF

5. Express the adjusted chained relatives as a percentage of their average to obtain seasonalindices.

6. Make sure that the sum of these indices is 400 for quarterly data and 1200 for monthly data.

Merits and Demerits

This method is less complicated than the ratio to moving average and the ratio to trend methods. However, this method is based upon the assumption of a linear trend which may not always hold true.

4.9 Summary

Business forecasting refers to the estimation of proposed business situations. It helps in proper planning, optimum utilization of resources, increase in productivity and systematic working in the business. There are various methods which may be applied for business forecasting viz. time series analysis, opinion poll method, survey method, regression analysis, econometric method etc.

Time series is the most prominently used method which helps in identifying the type of fluctuations in the business activities i.e. Secular, Cyclical, Seasonal or Irregular. There are various methods of working out these fluctuations. These are free hand method, statistical, algebraic and moving average method. The most widely applied method is moving average as it rules out the effect of extreme values in the data. These techniques are of immense use in business as it helps to reduce the risk and uncertainty associated with future.

4.10 Key Words

- **Business:** It refers to an organization or enterprising entity engaged in commercial, industrial or professional activities. A business can be a for-profit entity, such as a publicly-traded corporation, or a non-profit organization engaged in business activities.
- **Business Forecasting:** It is concerned with trying to reduce the uncertainty relating to future. Additional information is generated about the future that help the managers to assess the future consequences of existing decisions and to evaluate the consequences of alternatives.

- **Time Series:** A time series is a sequence of data points, measured typically at successive time instants spaced at uniform time intervals.
- **Variables:** Those entities whose value are not stable but keep on changing are called variables.

4.11 Self Assessment test

1. What do you understand by Business Forecasting? Explain the objectives underlying the business forecasting?
2. Discuss the methods of Business Forecasting?
3. Define Time Series? State the main components of Time Series?
4. Explain the preliminary adjustments required before analyzing time series data?
5. State the equations to find out the values of 'a' and 'b' constants in the trend equation:

   $Y_c = a + bX$
6. State the difference between seasonal variations and cyclical fluctuations?
7. Apply the semi average method for measuring trend to the following data:

Years	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1780	1981
Production	102	105	114	110	108	116	112	120	114	108	112	119

8. Calculate the trend value by 3 yearly moving averages from the following data:

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009
Sales	16	14	20	18	22	17	19	21	20

4.12 References

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Unit - 5 : Index Numbers

Structure Unit:

5.0 Objectives

After completing this unit, you will be able to:

- Understand that index numbers describe how much economic variables have changed over time;
- Point out various uses of index numbers;
- Know and avoid problems in constructing index numbers;
- Become familiar with the three principal types of index: price index, quantity index, and value index;
- Learn how to calculate various kinds of index numbers;
- Describe various limitations of index numbers.

5.1 Introduction

In our day to day life, things keep changing. The prices of various commodities vary at some rate over a period of time. In order to be update on such price changes, we need some methods to predict these. In business area for the budget purpose a manager may be interested to know how the raw material prices have increased over last one year. May be some changes in price; indicate the trend of increase on the regular basis, so that future planning could be more accurate. For such variations, we may analyze the degree of change in the form of Index Numbers.

Index numbers are indicators which reflect the relative changes in the level of a certain phenomenon in any given period (or over a specified period of time) called the ‘current period’ with respect to its values in some fixed period, called the ‘base period’ selected for comparison.

Index numbers are often known as the barometers of economic activity as they help to get an idea of the present day situation with regard to changes in production, consumption, exports and imports, national income, business level, cost of living, the price of a particular commodity or a group of commodities, Industrial or agricultural production, stocks and shares, sales and profits of a business house, volume of trade, factory production, wage structure of workers in various sectors, bank deposits, foreign exchange reserves, and so on.

Some of the important definitions of index numbers are given as under:

According to Wessell, Willet and Simone, “An index number is a special type of average that provides a measurement of relative changes from time to time or from place to place”.

According to Edgeworth, "Index number shows by its variations the changes in a magnitude which is not susceptible either of accurate measurement in itself or of direct valuation in practice".

According to Murray R. Spiegel, "An Index Number is a statistical measure designed to show changes in variable or a group of related variables with respect to time, geographic location or other characteristics."

According to Croxton and Lowden, "Index Numbers are devices for measuring differences in the magnitude of a group of related variables."

According to Karmal and Pollasek, "An index number is a device for comparing the general level of magnitude of a group of distinct, but related, variables in two or more situations".

According to Patterson, "In its simplest form, an index number is the ratio of two index numbers expressed as a percent. An index number is a statistical measure - a measure designed to show changes in one variable or in group of related variable over time or with respect of geographic location or other characteristics".

According to Dr. A. L. Bowley, "A series of index numbers is a series which reflects in its trend and fluctuations, the movements of some quantity to which it is related."

According to John I. Griffin, "An Index Number is a quantity which by reference to a base period shows by its variations the changes in the magnitude over a period of time. In general, Index Numbers are used to measure changes over time in magnitudes which are not capable of direct measurement."

Thus, it is apparent from above definitions that an index number is a statistical device which measures the extent to which a group of related variable changes over a period of time. Index number in fact relates a variable or group of variables in a given period to the same group of variables in some other period.

Some of the important characteristics of index numbers include the following:

- Index numbers are the specialized averages.
- Index numbers record the net changes in a group of related variables over a period of time.
- Index numbers measure changes not capable of direct measurement.
- Index numbers are for comparison.
- Index numbers are expressed in percentages.

Activity A:

1. "An Index number is a special type of average". Discuss.
2. According to you which definition of index number is more appropriate and why?

5.2 Uses of Index Numbers

The first index number was constructed by an Italian, Mr. Carli, in 1764 to compare the changes in price for the year 1750 (current year) with the price level in 1500 (base year) in order to study the effect of price level in Italy. Though originally designed to study the general level of prices or accordingly purchasing power of money, today index numbers are an important tool of economic analysis and they reveal the pulse of the economy.

Use of Index numbers is the most powerful tool in the hands of management, government officials and individuals to analyse the business and economic situations of a country. Some of the important uses or significance of index numbers to its users are listed as under:

1. Index Numbers Help in Formulating Policies and Decision Making: Formulation of good policies for the future depends upon past trends. Behaviour of the index numbers is studied carefully before making any policies. Index numbers of the data relating to prices, production, profits, imports and exports, personnel and financial matters are indispensable for any organisation in efficient planning and formulation of executive decisions.

For example the cost of living index numbers help the employers in deciding about the increase in dearness allowance of their employees or adjusting their salaries and wages in accordance with changes in their cost of living.

2. Reveal Trends and Tendencies: The index numbers study the relative changes in the level of a phenomena. So, they would disclose the general trend for a variable or group of variables in time series data. For example by examining the index numbers of production (industrial and agricultural), volume of trade, imports and exports etc. for the last few years, we can draw useful conclusions about the trend of production and business activity.

3. Index Numbers Measure the Purchasing Power of Money: Index numbers are helpful in finding out the intrinsic value of money as contrasted with its nominal worth. The cost of living index numbers determine whether the real wages are rising or falling, money wages remaining unchanged.

For example, suppose that the cost of living index for any year, say, 2010 for a particular class of people with 2001 as base year is 200. If a person belonging to that class gets **Rs.** 300 in 2001, then in order to maintain the same standard of living as in 2001 (other factors remaining constant) his salary in 2010 should be $\frac{200}{100} \times 300$ = Rs.600.

4. Aid in Deflation: Index numbers are very useful for deflating (or adjusting) the original data. In obtaining real income from inflated income, real wages from nominal wages, and real sales from nominal sales and so on, the index numbers are immensely useful.

5. They are Economic Barometers. As described above various index numbers are computed for different purposes, say employment, trade, transport, agriculture, industry, etc., and these are of immense value in dealing with different economic issues. Like barometers which are used in Physics and Chemistry to measure atmospheric pressure, index numbers are rightly termed as 'economic barometers' or 'barometers of economic activity' which measure the pressure of economic and business behaviour.

In the words of G. Simpson and F. Kafka, "Index numbers are today one of the most widely used statistical devices. They are used to take the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies."

6. Other Uses: Index numbers are thus, become indispensable tool in business planning and formulation of executive decisions of any concern and theses are today one of the most widely used statistical devices. Fairly good appraisal of trade and business activities of country can be done if careful study is carried out by constructing index. In brief the uses of index numbers are shown below:

- They measure the relative change
- They are of better comparison.
- They are wage adjuster.
- They compare standard of living.
- They are special type of averages.

Activity B:

1. "Index numbers are the devices for measuring differences in the magnitude of a group of related variables." Discuss this statement and point out the important uses of index numbers.

5.3 Types of Index Numbers

Index numbers may be divided into three categories:

(a) Price Index

(b) Quantity Index, and

(c) Value Index

(a) Price Index: It is the most commonly used index. It compares the prices of various commodities from one period to another. We may have steel price index, sugar price index, vegetables price index etc. For these purpose, a well known price index is called consumer price index (CPI), it is tabulated at regional or National level so as to establish the price levels of various consumer goods and services. This is an effective measure and useful index of cost of living. Index numbers are further divided into the following two categories:

(i) Wholesale Price Index Numbers: It reflect the changes in general price level of the country such as wholesale price index number prepared by government.

(ii) Retail Price Index Numbers: These index numbers show general changes in retail prices of various commodities such as consumption goods, stock and shares, bank deposits, consumer price index etc.

(b) Quantity Index: Quantity index numbers study the changes in the volume of goods produced, distributed or consumed, like the indices of agricultural production, industrial production, imports and exports, etc. These types of indices are useful for measuring the changes in level of physical output in an economy during some period compared with other period.

(c) Value Index: The last type of Index, the value index, measures changes in total monetary worth, during some period compared with other period. The value index numbers are intended to study the changes in the total value (quantity multiplied by price) of production.

5.4 Precautions in Construction of Index Numbers

Index numbers which are not properly compiled will, not only lead to wrong and fallacious conclusions but might also prove to be dangerous. So, the construction of index number requires a careful study of some aspects which often called 'precautions or problems'. The given below are some precautions:

1. Purpose of Index Numbers: It is essential to be clear about the purpose for which the index number is used. Every index number has its own particular uses. For example if it is used for measuring consumers' price, there is no need of including wholesale prices. Similarly, if it is employed for studying cost of living, there is no need of including the price of capital goods.

2. Selection of Base Period: Base period is the period against which comparisons are made. One has to be very careful in selecting a base period. If we select inappropriate base, then distortion can be large. Taking a year of large increase in prices due to abnormal causes may not be appropriate base. Also may be a large consumption of a commodity at the time of natural calamity may not reflect correct index level at the base year. Before selecting a base period the following points should be kept in mind:

(a) The base period should not get affected by extra-ordinary events like war, earthquakes, famines, booms, etc. it should be a normal one
(b) It should be relatively current i.e., it should not be too distant in the past because we are interested in the changes relating to the present period only.
(c) The base may be fixed, chain or average depending upon the purpose of constructing the index.

3. Selection of Commodities or Items: Only those commodities or items which are fairly represented and uniform quality should be selected for inclusion in the construction of index number. White selecting the sample the following points should be kept in mind:

• The commodities selected should be relevant to the purpose of the index.

• Select the adequate number of representative items from each group (Neither too small nor too large)

• Classify the whole relevant group of items or commodities into relatively homogeneous sub-groups.

4. Selection of Weights: 'Weights' imply the relative importance of the different variables. Due to inappropriate importance given to various factors in calculation of index, the calculated value will be found distorted and may not be the true representative of the decision variable. So proper weight should be assigned to different commodities with their relative importance in the group. Assigning weights can be done by (a) Implicit method; items are implicitly weighted, assumed to equal importance) and (b) Explicit method (weights are assigned either in quantity terms or in value terms). The choice of method of weighing depends on the purpose, scope, availability of data. Like in developing a composite index, such as the Consumer Price Index, we must consider changes in some variables to be more important than changes in others like, wheat should be given more importance compared to sugar.

5. Data Collection: The basic data used must be reliable, authentic and suitable for the purpose. Sufficient data is also required to arrive at a worthwhile information deduction. The source of data depends on their information requirements. In dealing with broad areas of national economy and the general level of business activity, publications such as the Federal Reserve Bulletin, Moody's, Monthly Labour Review, and the Consumer Price Index provide a wealth of data. Almost all government agencies distribute data about their activities, from which index numbers can be computed. Many financial newspapers and magazines provide information from which index numbers can be computed. When you read these sources, you will find that many of them use index numbers themselves.

6. Selection of Average: Averages play a vital role in arriving at a single index number summarizing a large volume of information. Arithmetic mean and geometric mean are used in it construction but theoretically, the geometric mean is preferred because it is less susceptible to variation; it gives equal weight to equal ratio of change.

7. Price Collection: After selecting the items, the next problem is to collect their prices. The price of a commodity varies from place to place and even from shop to shop in the same market. So, it is very difficult to consider and compile prices from every market, from every shop and for all periods. Therefore, we should select, a sample market, which are well known for trading in a particular commodity and also collect the data of price for that commodity from the agencies such as the Chambers of Commerce, News Correspondents etc. and further we should compare for validity and reliability.

8. Selection of Appropriate Formula: A large number of formulas have been devised for constructing the index numbers. A decision has, therefore, to be made as to which formula is the most suitable for the purpose. The choice of the formula depends upon the availability of the data regarding the prices and quantities of the selected commodities in the base and/or current year.

Activity C:

1. What is an Index number? Why index numbers are called "Economic barometers"?
2. Analyze the problems in the construction of index numbers and comment.

5.5 Notations and Methods of Constructing Index Numbers

Notations:

Base Year: The year selected for comparison i.e. the year with reference to which comparisons are made. It is denoted by '0'.

Current Year: The year for which comparisons are sought or required.

P_0 is price of a commodity in the base year,
P_1 is price of a commodity in the current year,
q_0 is quantity of a commodity in the base year,
q_1 is quantity of a commodity in the current year,
W is weight assigned to a commodity according to its relative importance in group,
P_{01} is price index number for the current year,
P_{10} is price index number for the base year,
Q_{01} is quantity index number for the current year, and
q_{10} is quantity index number for the base year

Methods of Constructing Index Numbers

(I) Price Index Numbers

Methods of constructing index numbers can broadly be divided into two classes namely:

(A) Un-weighted Index Numbers, and
(B) Weighted Index Numbers.

In case of un-weighted indices, weights are not assigned, whereas in the weighted indices weights are assigned to the various items. Each of these types may be further classified under two heads:

(i) Aggregate of Prices Method, and
(ii) Average of Price Relatives Method.

The following chart illustrates the various methods:

Index Numbers

- Un-weighted
  - Simple Aggregate of prices
  - Simple Average of price relatives
- Weighted
  - Weighted Aggregate of prices
  - Weighted Average of price relatives

(A) Un-weighted Index Numbers:

(i) Simple (Un-weighted) Aggregate Method: This is the simplest methods of constructing index numbers and consists in expressing the total price, i.e., aggregate of prices (of all the selected commodities) in the current year as a percentage of the aggregate of prices in the base year. Thus, the price index for the current year w.r.t. the base year is given by:

$$P_{10} = \frac{\Sigma p_1}{\Sigma p_0} x 100$$

Where Σp_1 is the aggregate of prices in the current year and Σp_0 is the aggregate of prices in the base year. This method suffers from a drawback that equal weight is given to all the items irrespective of their relative importance.

Illustration: 1

From the following data calculate Index Number by simple aggregate method.

Commodity	:	A	B	C	D
Price in 1990 (Rs.)	:	162	256	257	132
Price in 1991 (Rs.)	:	171	164	189	145

Solution:

Computation of Price Index Number

	Price (in Rupees)	
	1990 (P_0)	**1991(p_1)**
A	162	171
B	256	164
C	257	189
D	132	145
Total	**Σp_0=807**	**Σp_1=669**

The price index number using simple aggregate method is given by:

$$P_{10} = \frac{\Sigma p_1}{\Sigma p_0} X\ 100$$

$$= \frac{669}{807} X\ 100$$

$= 82.90$

Illustration: 2

Following prices are indicated for 2005 (base year) and for 2010 (the current year). Calculate the un-weighted aggregates price index for the data

Variables	Prices	
	2005	2010
Tomatoes (per kg.)	Rs. 15.00	Rs. 19.00
Egg (per dozen)	Rs. 20.00	Rs. 24.00
Petrol (Per Liter)	Rs. 22.50	Rs. 30.70
Juices (Per Liter)	Rs. 61.00	Rs. 69.00

Solution:

For computing price index (un-weighted) we have

Σp_1 = Rs. 19+24+30.70+69 = Rs. 142.70
Σp_0 = Rs. 15+20+22.50+61=Rs. 118.50

Un-weighted aggregates price index

$$= \frac{142.70}{118.50} \text{ x } 100 = 120$$

(ii) Simple Average of Price Relatives: In order to calculate, we can compare the ratio of current prices to the base prices and then the index is calculated by multiplying the rate by 100. We, then, take the average of all the ratios summed up together. The general relationship now undergoes the change as follows:

Un-weighted average of relative price index

$$= \frac{\Sigma \frac{P_1}{P_0}}{n} x100$$

The simple average of price relatives method is superior to the simple aggregate of prices method in two respects:

(i) Since we are comparing price per liter with price per liter, and price per kilogram with price per kilogram the concealed weight due to use of different units is completely removed.
(ii) The index is not influenced by extreme items as, equal importance is given to all items.

The greatest drawback of unweighted indices is that equal importance or weight is given to all items included in the index number which is not proper. As such, unweighted indices are of little use in practice.

Illustration: 3

From the following data, construct index number by simple average of price relatives using arithmetic mean

Commodity	1998 Price (Rs.)	2001 (Price (Rs.)
Wheat	800/quintal	1000/quintal
Rice	15/Kg.	19/Kg.
Milk	12/Liter	15/Liter
Eggs	10/Dozen	12/Dozen
Sugar	14/Kg.	18/Kg.

Solution:

Commodity	P_0	P_1	P (Price relative)		
Wheat	800	1000	1000 x 100/800	=	125
Rice	15	19	19 x 100/15	=	126.67
Milk	12	15	15 x 100/12	=	125
Eggs	10	12	12 x 100/10	=	120
Sugar	14	18	18 x 100/14	=	128.57
					625.24

Using simple Arithmetic Mean:

Average of relative price index

$$= \frac{\sum \frac{P_1}{P_0}}{n} x100$$

$$= \frac{625.24}{5} = 125.05$$

(B) Weighted Index Numbers:

The purpose of weighting is to make the index numbers more representative and to give more importance to them. Weighted index numbers are of two types. They are:

(i) Weighted Aggregate Index Numbers. According to this method, prices themselves are weighted by quantities; i.e., p x q. Thus physical quantities are used as weights. Here are various methods of assigning weights, and thus various formulas have been formed for the construction of index numbers. Some of the important formulae are given below:

1. Laspeyre's Method
2. Paasche's Method
3. Dorbish and Bowley's Method
4. Marshall-Edgeworth Method
5. Kelly's Method, and
6. Fisher's Ideal Method

1. Laspeyre's Method. In this method, base year quantities are taken as weights. The formula for constructing the index is:

$$P_{01} = \frac{\sum P_1 q_0}{\sum P_0 q_0} X\ 100$$

Where P_1 = Price in the current year

P_0 = Price in the base year

q_0 = Quantity in the base year

According to this method, the index number for each year is obtained in three steps:

(a) The price of each commodity in each year is multiplied by the base year quantity of that commodity. For the base year, each product is symbolised by P_0q_0, and for the current year by P_1q_0.

(b) The products for each year are totaled and $\sum P_1q_0$ and $\sum P_0q_0$ are obtained.

(c) $\sum P_1q_0$ is divided by $\sum P_0q_0$ and the quotient is multiplied by 100 to obtain the index.

Laspeyre's index is very widely used in practice. It tells us about the change in the aggregate value of base period list of goods when valued at given period price. However, this index has one drawback. It does not take into consideration the changes in the consumption pattern that take place with the passage of time.

2. **Paasche's Method**. In this method, the current year quantities are taken as weights: symbolically,

$$P_{01(Pa)} = \frac{\sum p_1 q_1}{\sum p_0 q_1} x100$$

According to this method, the index number for each year is obtained in following steps:

1. Multiply current year prices of various commodities with current year weights and obtain P_1q_1.
2. Multiply the base year prices of various commodities with the current year weights and obtain P_0q_1.
3. $\sum P_1q_1$ is divided by $\sum P_0q_1$ and the quotient is multiplied by 100 to obtain the index

Although this method takes into consideration the changes in the consumption pattern, the need for collecting data regarding quantities for each year or each period makes the method very expensive. Hence, where the number of commodities is large, Paasche's method is not used in practice.

3. Bowley Dorbish Method. This is an index number got by the arithmetic mean of Laspeyre's and Paasche's methods; symbolically (This method takes into account both the current and the base periods). Symbolically

$$P_{01(B)} = \frac{\frac{\sum p_1 q_0}{\sum p_0 q_0} + \frac{\sum p_1 q_1}{\sum p_0 q_1}}{2} x100 = \frac{L+P}{2}$$

L = Laspeyre's method
P = Paasche's method

4. Marshall-Edgeworth Method. In this method, the totals of base year and current year quantities are taken as weights. The formula for constructing the index is:

$$P_{01} = \frac{\sum P_1(q_0 + q_1)}{\sum P_0(q_0 + q_1)} x100$$

$$or\ P_{01} = \frac{\sum p_1 q_0}{\sum P_0 q_0} + \frac{\sum P_1 q_1}{\sum p_0 q_1} x100$$

5. Kelly's Price Index or Fixed Weights Index. This formula, named after Truman L. Kelly, requires the weights to be fixed for all periods and is also sometimes known as *aggregative index with fixed weights* and is given by the formula:

$$P_{01} = \frac{\sum p_1 q}{\sum P_0 q} X\ 100$$

Where the weights are the quantities (q) which may refer to some period (not necessarily the base year or the current year) and are kept constant for all periods. The average (A.M. or G.M.) of the quantities consumed of two, three or more years may be used as weights.

Kelly's fixed base index has a distinct advantage over Laspeyre's index because unlike Laspeyre's index the change in the base year does not necessitate a corresponding change in the weights which can be kept constant until new data become available to revise the index. As such, currently this index is finding great favour and becoming quite popular.

6. Fisher's Ideal Index. This method is the geometric mean of Laspeyre's and Paasche's indices. The formula for constructing the index is :

$$P_{01} = \sqrt{\frac{\sum P_1 q_0}{\sum P_0 q_0} x \frac{\sum P_1 q_1}{\sum P_0 q_1}} \times\ 100$$

Because of the following reasons, Fisher's formula is known as 'ideal':

- It takes into account prices and quantities of both current year as well as base year.
- It uses geometric mean which, theoretically, is the best average for constructing index numbers.
- It satisfies both time reversal test and the factor reversal test.
- It is free from bias. The weight biases embodied in Laspeyre's and Paasche's methods are crossed geometrically and thus eliminated completely.

Illustration: 4

By using Laspeyre's method, calculate the weighted price index for the year 2010 when the given data indicates the prices and consumption levels of various commodities.

Commodities	*Base Price (2007)*	*Current Price (2010)*	*Average quantity consumed (2007)*
Potatos (per kg)	Rs.5.10	Rs.4.50	4000 Kgs
Milk (per litre)	Rs.14.00	Rs.17.00	800 litres
Eggs (per doz)	Rs.21.00	Rs.24.00	2000 dozens
Bread (per loaf)	Rs.17.50	Rs.19.00	350 loaves

Solution:

For working out Laspeyre's Price Index, we prepare the table as follows:

Commodities	*Price in 2007*	*Price in 2010*	*Quantity in 2007*		
	(P_0)	*(P_1)*	*(Q_0)*	*P_0Q_0*	*P_1Q_0*
Potatos (per kg)	Rs.5.10	Rs.4.50	4000 Kgs	20,400	18,000
Milk (per litre)	Rs.14.00	Rs.17.00	800 litres	11,200	13,600
Eggs (per doz)	Rs.21.00	Rs.24.00	2000 dozens	42,000	48,000
Bread (per loaf)	Rs.17.50	Rs.19.00	350 loaves	6,125	6,650

From the above calculations

ΣP_0Q_0 = Rs. 79,725

ΣP_1Q_0 = Rs. 86,250

Laspeyre's Price Index = $= \frac{\Sigma P_1Q_0}{\Sigma P_0Q_0}$ x 100

$$= \frac{86,250}{79,725} x100$$

= 108

Illustration: 5

From the following data, construct the Laspeyre's, Paasche's and Fisher's indices of prices:

Item	*Base Year*		*Current Year*	
	P_0	*q_0*	*P_1*	*q_1*
A	4	20	10	15
B	8	4	16	5
C	2	10	4	12
D	10	5	20	6

Solution:

Calculation of Price Index Numbers

Item	*Base Year*		*Current Year*					
	P_0	q_0	P_1	q_1	P_0q_0	P_1q_0	P_0q_1	P_1q_1
A	4	20	10	15	80	200	60	150
B	8	4	16	5	32	64	40	80
C	2	10	4	12	20	40	24	48
D	10	5	20	6	50	100	60	120
Total					**ΣP_0q_0=182**	**ΣP_1q_0=404**	**ΣP_0q_1=184**	**ΣP_1q_1=398**

Laspeyre's Price Index (L) $= \frac{\sum P_1q_0}{\sum P_0q_0} \times 100 = \frac{404}{182} x100 = 221.98$

Paasche's Method (P) $= \frac{\sum P_1q_1}{\sum P_0q_1} \times 100 = \frac{398}{184} x100 = 216.30$

Fisher's Ideal Index $= \sqrt{L\ x\ P}$

$= \sqrt{221.98\ x\ \ 216.3} = 219.12$

Illustration: 6

From the following data, calculate the price index numbers for 2008 with 2000 as base by:

(a) Laspeyre's method

(b) Paasche's method

(c) Bowley method

(d) Marshall-Edgeworth method

(e) Fisher's Ideal method

Item	*2000*		*2008*	
	Price (Rs.)	*Quantity (unit)*	*Price (Rs.)*	*Quantity (unit)*
Maize	70	28	140	21.0
Millet	175	35	210	17.5
Sugar	140	52.5	175	52.5
Coconut	70	70.0	70	87.5

Solution:

Calculation of Price Index Numbers

Item	P_0	q_0	P_1	q_1	P_1q_0	P_0q_0	P_1q_1	P_0q_1
Maize	70	28	140	21.0	3920	1960	2940	1470
Millet	175	35	210	17.5	7350	6125	3675	3062.5
Sugar	140	52.5	175	52.5	9187.5	7350	9187.5	7350
Coconut	70	70	70	87.5	4900	4900	6125	6125
					25357.5	**20335**	**21927.5**	**18007.5**

(a) Laspeyre's method $= P_{01} = \frac{\sum p_1q_0}{\sum p_0q_0} x100 = \frac{25357.5}{20335} x100 = 124.69$

(b) Paasche's method $= P_{01} = \frac{\sum p_1q_1}{\sum p_0q_1} x100 = \frac{21927.5}{18007.5} x100 = 121.76$

(c) Bowley method $= P_{01} = \frac{L+P}{2} = \frac{124.69 + 121.76}{2} = 123.22$

(d) Marshall-Edgeworth method $= P_{01} = \frac{\sum p_1 q_0 + \sum p_1 q_1}{\sum p_0 q_0 + \sum p_0 q_1} x100$

$$= \frac{25357.5 + 21927.5}{20335 + 18007.5} x100 = 123.32$$

(e) Fisher's Ideal method $= P_{01} = \sqrt{L \, x \, P} \, x100$

$$= \sqrt{124.70 \, x \, 121.77} \, x100 = 123.23$$

Activity D:

From the data given below, construct index number of prices for 2006 with 2000 as base, using

(i) Laspeyre's method,

(ii) Paasche's method,

(iii) Bowley-Drobisch method,

(iv) Marshall - Edgeworth method, and

(v) Fisher's ideal formula.

Commodity	2000		2006	
	Price per unit	Expenditure in rupees	Price per unit	Expenditure in rupees
A	2	10	4	16
B	3	12	6	18
C	1	8	2	14
D	4	20	8	32

(ii) Weighted Average of Price Relatives: This method is similar to the simple average of price relatives method with the fundamental difference that explicit weights are assigned to each commodity included in the index. Since price relatives are in percentages, the weights used are value weights. The following steps are taken in the construction of weighted average of price relatives index:

(i) Calculate the price relatives, $\left[\frac{P_1}{P_0} x100\right]$, for each commodity

(ii) Determine the value weight of each commodity in the group by multiplying its price in base year by its quantity in the base year, i.e., calculate P_0q_0 for each commodity. If, however, current year quantities are given, then the weights shall be represented by P_1q_1.

(iii) Multiply the price relative of each commodity by its value weight as calculated in above (ii).

(iv) Total the products obtained under (iii) above.

(v) Divide the total (iv) above by the total of the value weights. Symbolically index number obtained by the method of weighted average of price relatives is:

$$P_{01} = \frac{\sum \left[\left(\frac{P_1}{P_0} x100\right) P_0 q_0\right]}{\sum P_0 q_0} \; or \; \frac{\sum PV}{\sum V}$$

Illustration: 7

Based on the data given in illustration 4 calculate the weighted average of relatives index.

Solution:

Calculation of Weighted Average of Relatives Index

Commodities	*(P_0)*	*(P_1)*	*(Q_0)*	$\left[\frac{P_1}{P_0}\right] x100$	P_0Q_0	$\left[\frac{P_1}{P_0}\right] x100\ (P_0Q_0)$
Potatos	Rs.5.10	Rs.4.50	4000	88.23	20,400	17,99,892
Milk	Rs.14.00	Rs.19.00	800	135.71	11,200	15,19,952
Eggs	Rs.21.00	Rs.24.00	2000	114.28	42,000	47,99,760
Bread	Rs.17.50	Rs.19.00	350	108.57	6,125	6,64,991

From the above calculations, we obtain the values

ΣP_0Q_0 = Rs.79,725

and $\sum\left[\left(\frac{P_1}{P_0}\right)x100(P_0Q_0)\right] = 87{,}84{,}595$

$\therefore$ Weighted Average of the relatives index

$$= \frac{\Sigma\left[\left(\frac{P_1}{P_0}\right)x100(P_0Q_0)\right]}{\Sigma P_0Q_0}$$

$$= \frac{87{,}87{,}595}{79{,}725} = 110$$

(II) Quantity Index Numbers:

A quantity index number is a statistical device which measures changes in quantities in current year as compared to base year. Quantity index numbers reflect the relative changes in the quantity or volume of goods produced, consumed, marketed or distributed in any given year *w.r.t.* to some base year.

The formulae for calculating the quantity index numbers can be directly written from price index numbers simply by interchanging the role of price and quantity.

Thus quantity index by different methods is:

(a) Laspeyre's method = $Q_{01} = \frac{\sum q_1p_0}{\sum q_0p_0}x100$

(b) Paasche's method = $Q_{01} = \frac{\sum q_1p_1}{\sum q_0p_1}x100$

(c) Fisher's Ideal method = $Q_{01} = \sqrt{\frac{\sum q_1P_0}{\sum q_0P_0}x\frac{\sum q_1P_1}{\sum q_0P_1}}\ x100$

Illustration: 8. Compute quantity index for the year 1992 with base 1990=100, for the following data, using (i) Laspeyre's method (ii) Paasche's method. (iii) Fisher's ideal formula.

Item	Price		Quantities	
	1990	***1992***	***1990***	***1992***
A	5.00	6.50	5	7
B	7.75	8.80	6	10
C	9.63	7.75	4	6
D	12.50	12.75	9	9

Solution:

Computation of Quantity Index

Commodity	P_0	q_0	P_1	q_1	q_0P_0	q_0P_1	q_1P_0	q_1P_1
A	5.00	5	6.50	7	25.00	32.50	35.00	45.50
B	7.75	6	8.80	10	46.50	52.80	77.50	88.00
C	9.63	4	7.75	6	38.52	31.00	57.78	46.50
D	12.50	9	12.75	9	112.50	114.75	112.50	114.75
					$\Sigma q_0P_0 =$ **222.52**	$\Sigma q_0P_1 =$ **231.05**	$\Sigma q_1P_0 =$ **282.78**	$\Sigma q_1P_1 =$ **294.75**

(i) Laspeyre's quantity index or Q_{01} $= \frac{\sum q_1 P_0}{\sum q_0 P_0} x100$

$$= \frac{282.78}{222.52} x100 = 127.08$$

(ii) Paasche's quantity index or Q_{01} $= \frac{\sum q_1 P_1}{\sum q_0 P_1} x100$

$$= \frac{294.75}{231.05} x100 = 127.57$$

(iii) Fisher's quantity index or Q_{01} $= \sqrt{\frac{\sum q_1 P_0}{\sum q_0 P_0} x \frac{\sum q_1 P_1}{\sum q_0 P_1}}\, x100$

$$= \sqrt{\frac{282.78}{222.52} x \frac{294.75}{231.05}}\, x100$$

= 1.273 x 100
= 127.3

(III) Value Index Numbers:

These index measures the changes in the total value of the variable. Since value is a combination of price and quantity, it can be called a composite index. The only negative issue is that composite value index does not distinguish the variations in individual values of price or quantity separately. Value index numbers are obtained on expressing the total value (or expenditure) in any given year as a percentage of the same in the base year. Symbolically, we write

$$V_{01} = \frac{Total\ value\ in\ current\ year}{Total\ value\ in\ base\ year} x100 \quad \Rightarrow \quad V_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_0} x100$$

The value index number based on the information given in illustration 8 can be calculated as under:

Value Index number $= V_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_0} x100 = \frac{294.75}{222.52} x100 = 132.46$

5.6 Test of Consistency

As there are several formulae for constructing index numbers the problem is to select the most appropriate formula in a given situation. Prof. Irving Fisher has suggested two tests for selecting an appropriate formula.

These are:

(A) Time Reversal Test, and

(B) Factor Reversal Test

(A) Time Reversal Test:

This test requires that the formulae for calculating an index number should give consistent results in both the directions, i.e. forward and backward. Or in other words, the index of period 1 with period 0 base should be reciprocal of the index of period 0 with period 1 as base i.e. $P_{01} = 1/P_{10}$ or p_{01} x $P_{10} = 1$.

This test is satisfied by the Fisher's Ideal Index.

We can write

$$P_{01}^{F1} = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} \ (dropping\ 100)$$

$$P_{10} = \sqrt{\frac{\sum p_0q_1}{\sum p_1q_1} x \frac{\sum p_0q_0}{\sum p_1q_0}}$$

$$\therefore \quad P_{01}^F x P_{10}^F = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum p_0q_1}{\sum p_1q_1} x \frac{\sum p_0q_0}{\sum p_1q_0}} = \sqrt{1} = 1$$

(B) Factor Reversal Test

This test requires that the product of price index and the corresponding quantity index numbers should be equal to the value index number i.e. $P_{01}xQ_{01}=V_{01}$.

This test is also satisfied by the Fisher's ideal index.

We can write

$$P_{01}^{F1} = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}}$$

$$\text{and } Q_{01}^{F1} = \sqrt{\frac{\sum q_1P_0}{\sum q_0P_0} x \frac{\sum q_1p_1}{\sum q_0p_1}}$$

$$\therefore \quad P_{01}^F x q_{01}^F = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum q_1p_0}{\sum q_0p_0} x \frac{\sum q_1p_1}{\sum q_0p_1}} = \sqrt{\left(\frac{\sum p_1q_1}{\sum p_0q_0}\right)^2} = \frac{\sum p_1q_1}{\sum p_0q_1} = V_{01}$$

Illustration: 9

Compute Index Number, using Fishers Ideal formula and show that it satisfies time-reversal test and factor-reversal test.

	Quantity	*Base Year Price*	*Quantity*	*Current year Price*
A	12	10	15	12
B	15	7	20	5
C	24	5	20	9
D	5	16	5	14

Solution:

Computation of Index Number

Commodity	q_0	p_0	q_1	p_1	p_1q_0	p_0q_0	p_1q_1	p_0q_1
A	12	10	15	12	144	120	180	150
B	15	7	20	5	75	105	100	140
C	24	5	20	9	216	120	180	100
D	5	16	5	14	70	80	70	80
					$\Sigma p_1q_0 = 505$	$\Sigma p_0q_0 = 425$	$\Sigma p_1q_1 = 530$	$\Sigma p_0q_1 = 470$

$$P_{01} = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} \text{ x } 100$$

$$= \sqrt{\frac{505}{425} x \frac{530}{470}} \text{ x } 100$$

$$= \sqrt{1.188 \; x \; 1.128} \; x \; 100$$

$$= \sqrt{1.340} \; x \; 100 = 1.158 \; x \; 100$$

$$= 115.8$$

(a) Time-Reversal Test

Time Reversal Test is satisfied when P_{01} x $P_{10} = 1$

$$P_{10} = \sqrt{\frac{\sum p_0q_1}{\sum p_1q_1} x \frac{\sum p_0q_0}{\sum p_1q_0}}$$

$$= \sqrt{\frac{470}{530} x \frac{425}{505}}$$

$$P_{01} \text{ x } P_{10} = \sqrt{\frac{505}{425} \; x \; \frac{530}{470} \; x \; \frac{470}{530} x \frac{425}{505}} = \sqrt{1}$$

= 1

(b) Factor-Reversal Test

Factor-Reversal test is satisfied when

$$P_{01} = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}}$$

$$Q_{01} = \sqrt{\frac{\sum q_1p_0}{\sum q_0p_0} x \frac{\sum q_1p_1}{\sum q_0p_1}}$$

$$P_{01} \text{x} Q_{01} = \sqrt{\frac{505}{425} \; x \; \frac{530}{470} \; x \; \frac{470}{425} x \frac{530}{505}}$$

$$= \sqrt{\frac{530}{425}} i.e., \frac{\sum p_1q_1}{\sum p_0q_0}$$

$$P_{01} \text{x} Q_{01} = \frac{\sum p_1q_1}{\sum p_0q_0}$$

Hence the given data satisfies the time-reversal test and factor-reversal test.

Activity E:

1. What are the tests of a good index number? Define Fisher's Ideal index number and show that it satisfies all these tests.
2. State whether the following statements are 'True' or 'Untrue':
(i) Arithmetic mean is the most appropriate average for constructing the index numbers.
(ii) Paasche's Index number is based on base year quantity.
(iii) Fisher's Index Number is an Ideal Index Number.
(iv) Fisher's Index Numbers is the arithmetic average of Laspeyre's and Paasche's Index Numbers.
(v) Time reversal test is satisfied by both formulas Fisher and Kelly's.

5.7 Fixed and Chain Base Index Numbers

Fixed Base Index Numbers:

When the comparison of (prices or quantities etc.) various periods are done with reference to a particular or fixed period, we get an index number series with fixed base.

Chain Base Index Numbers:

The main problem with a fixed base series arises when the current year becomes too far off from the base year. In such a situation, it may happen that the commodities which used to be very important in the base year are no longer so in current year. Furthermore, certain new commodities might be in use while some old commodities are dropped in current year. This problem is often solved by constructing Chain Base Index Numbers. A chain base index number is an index number with previous year as base.

Differences between Chain Base Method and Fixed Base Method

S. No.	***Chain Base***	***Fixed Base***
1.	The base year changes.	The base year does not change.
2.	Here the link relative method is used.	No such link relative method is used.
3.	Introduction and deletion of items are easy to calculate, without recalculation of the entire series.	Any change in the commodities, will involve the entire index number to be recast.
4.	The calculations are tedious	The calculations are simple.
5.	It is difficult to understand.	It is simple to understand.
6.	It cannot be computed if data for any one year are missing.	There is no such problem.
7.	It is suitable for short period only.	It is suitable for long periods only.
8.	Weights can be adjusted as frequently as possible.	Weights cannot be adjusted so frequently.
9.	Index number is wrong if an error is committed in the calculation of any link index number.	This is not so, the error is confined to the index of that year only.

Illustration: 10. From the following data relating to the wholesale prices of wheat for six years, construct index numbers using (a) 1990 as base, and (b) by chain base method.

Year	*Price (per quintal) Rs.*	*Year*	*Price (per quintal) Rs.*
1990	100	**1993**	130
1991	120	**1994**	140
1992	125	**1995**	150

Solution:

(a) Computation of Index numbers with 1980 as base:

Year	*Price of wheat*	*Index Number 1980 = 100*	*Year*	*Price of wheat*	*Index Number 1980 = 100*
1990	100	100	**1993**	130	$\frac{130}{100}x100 = 130$
1991	120	$\frac{120}{100}x100 = 120$	**1994**	140	$\frac{140}{100}x100 = 140$
1992	125	$\frac{125}{100}x100 = 125$	**1955**	150	$\frac{150}{100}x100 = 150$

(b) Construction of Link Relative Indices

Year	*Price of wheat*	*Link Relative Index*	*Year*	*Price of wheat*	*Link Relative Index*
1990	100	100	**1993**	130	$\frac{130}{125}x100 = 104$
1991	120	$\frac{120}{100}x100 = 120$	**1994**	140	$\frac{140}{130}x100 = 107.692$
1992	125	$\frac{125}{120}x100 = 104.167$	**1995**	150	$\frac{150}{140}x100 = 107.14$

Conversion of Link Relatives into Chain Relatives:

Chain relatives or chain indices can be obtained either directly or by converting link relatives into chain relatives with the help of the following formula:

$$\text{Chain relative for current year} = \frac{\text{Link Relative for the current year} \ X \ \text{Chain relative for the previous year}}{100}$$

Taking the data from illustration 10, we can show the method of conversion as follows:

Year	*Price of wheat*	*Link relative*	*Chain relative*
1990	100	100.00	100
1991	120	120.00	$\frac{120\,x\,100}{100} = 120$
1992	125	104.167	$\frac{104.167\,x\,120}{100} = 125$
1993	130	104.00	$\frac{104\,x\,125}{100} = 130$
1944	140	107.692	$\frac{107.692\,x\,130}{100} = 140$
1955	150	107.14	$\frac{107.14\,x\,140}{100} = 150$

5.8 Base Shifting, Spicing and Deflating the Index Numbers

Base Shifting:

Sometimes it becomes necessary to shift the base from one period to another. This becomes necessary either because the previous base has become too old and has become useless for comparison purposes or because comparison has to be made with another series of index numbers having different base period.

The following formula must be used in this method of base shifting:

Index Number (based on New Base Year)

$$= \frac{Current\ year's\ old\ index\ number}{New\ base\ year's\ old\ index\ number} x\ 100$$

Illustration: 11

Shift the base of the following series to 1997.

Year	1995	1996	1997	1998	1999	2000
Index No.	125	155	185	220	265	320

Solution:

To shift the base at 1997, we multiply every index number by 100/185.

Year	1995	1996	1997	1998	1999	2000
Index No.	67.6	83.8	100	118.9	143.2	173.0

Splicing Two Index Number Series:

The statistical method connects an old index number series with a revised series in order to make the series continuous is called splicing. The articles which are included in an index number may become out of fashion or go out of the market. New articles come into the market, for which relative importance may also change. So it is necessary to include the articles in the index number. The old series of index number is discontinued and we must construct a new series and must take the year of discontinuation as the first base.

Thus we connect the new set of index with the old discontinued one. The formula is:

$$= \frac{Index\ no.of\ current\ year\ x\ old\ Index\ No.of\ New\ base\ year}{100}$$

Illustration: 12

Two sets of Indices, one with 1986 as base and the other with 1994 as base are given below :

(a) Year	*Index Numbers*	*(b) Year*	*Index Number*
1986	100	**1994**	100
1987	110	**1995**	105
1988	120	**1996**	90
1989	190	**1997**	95
1990	300	**1998**	102
1991	330	**1999**	110
1992	360	**2000**	96
1993	390		
1994	400		

The Index (a) with 1986 base was discontinued in 1994. You are required to splice the second index number (b) with 1994 base to the first index number.

Solution:

1986 1994

Splicing of Index Numbers

Year	*Index Number (a) with 1976 as base*	*Index Number (b) with 1984 as base*	*Index Number (b) spliced to (a) with 1976 as base*
1986	100		
1987	110		
1988	120		
1989	190		
1990	300		
1991	330		
1992	360		
1993	390		
1994	400	100	$100\ x \frac{400}{100} = 400$
1995		105	$105\ x \frac{400}{100} = 420$
1996		90	$90\ x \frac{400}{100} = 360$
1997		95	$95\ x \frac{400}{100} = 380$
1998		102	$102\ x \frac{400}{100} = 408$
1999		110	$110\ x \frac{400}{100} = 440$
2000		96	$96\ x \frac{400}{100} = 384$

Deflating:

Deflating is the process of making allowances for the effect of changing price levels. With increasing price levels, the purchasing power of money is reduced. As a result, the real wage figures are reduced and the real wages become less than the money wages. To get the real wage figure, the money wage figure may be reduced to the extent the price level has raised. The process of calculating the real wages by applying index numbers to the money wages so as to allow for the change in the price level is called deflating. Thus, deflating is the process by which a series of money wages or incomes can be corrected for price changes to find out the level of real wages or incomes. This is done with the help of the following formula:

Real Wage $= \frac{Money\ Wage}{Price\ Index}\ x\ 100, and$

Real Wage Index $= \frac{Real\ wages\ for\ the\ year}{Real\ wages\ for\ the\ Base\ year}\ x\ 100$

Illustration: 13

Given the following data:

Year	*Weekly take-home pay (wages)*	*Consumer Price Index*
1991	109.5	112.8
1992	112.2	118.2
1993	116.4	127.4
1994	125.08	138.2
1995	135.4	143.5
1996	138.1	149.8

(1) What was the real average weekly wage for each year?

(2) In which year did the employee have the greatest buying power?

(3) What percentage increase in the weekly wages for the year 1996 is required, if any, to provide the same buying power that the employees enjoyed in the year in which they had the highest real wages?

Solution:

Calculation of Real Wages

Year	*Weekly take-home pay (Rs.)*	*Consumer price Index*	*Real Wages*
1991	109.5	112.8	$\frac{109.5}{112.8} x\ 100 = 97.07$
1992	112.2	118.2	$\frac{112.2}{118.2} x\ 100 = 94.92$
1993	116.4	127.4	$\frac{116.4}{127.4} x\ 100 = 91.37$
1994	125.08	138.2	$\frac{125.08}{138.2} x\ 100 = 90.51$
1995	135.4	143.5	$\frac{135.4}{143.5} x\ 100 = 94.36$
1996	138.1	149.8	$\frac{138.1}{149.8} x\ 100 = 92.19$

(1) Real average weekly wage can be obtained by the formula:

Real Wage $= \frac{Money\ Wage}{Price\ Index} x\ 100$

(2) The employee had the greatest buying power in 1991 as the real wage was maximum in 1991.

(3) Absolute difference = 97.07 - 92.19 = + 4.88

5.9 Limitations of Index Numbers

Even though index numbers are very important in business and economic activities, they have their own limitations; they are:

1. The index numbers are only approximate indicators because their construction based on the sample data, and may not exactly represent the true changes in relative level of a phenomenon.

2. An index number does not take into account the quality of items.

3. Likelihood of error is possible at each stage of construction of index numbers, viz., (i) selection of commodities, (ii) selection of the base period, (iii) collection of data - prices and quantities of commodities, (iv) choice of formula - the procedure of weight age to be given.

4. Index number is an average and as such it suffers from all the limitations of an average.

5. Consumption is the result of taste, custom, attitude, etc., which are dynamic.

6. There is no unique index number that is acceptable to all.

7. There may be possibility of manipulation of the base year, price, commodities and quantity quotations in order to get the required results by the selfish persons.

5.10 Summary

Index numbers help to get an idea of the present day situation with regard to changes in production, consumption, exports and imports, national income, business level, cost of living, the price of a particular commodity or a group of commodities, Industrial or agricultural production, stocks and shares, sales and profits of a business house, volume of trade, factory production, wage structure of workers in various sectors, bank deposits, foreign exchange reserves, and so on.

Index numbers help in formulating policies and decision making, reveal trends and tendencies, measure the purchasing power of money, aid in deflation, and work like economic barometers. Index numbers may be divided into three categories price index, quantity index, and value index.

Construction of index numbers requires a careful study of some aspects like purpose of index numbers, selection of base period, selection of commodities or items, selection of weights, collection of data, selection of average, price collection, and selection of appropriate formula to construct the index numbers. Methods of constructing index numbers can broadly be divided into two classes namely: un-weighted index numbers, and weighted index numbers. In case of un-weighted indices, weights are not assigned, whereas in the weighted indices weights are assigned to the various items. Each of these types may be further classified under two heads: aggregate of prices method, and average of price relatives method.

Even though index numbers are very important in business and economic activities, they have their own limitations; like index numbers are based on the sample data, and may not exactly represent the true changes in relative level of a phenomena, error is possible at each stage of construction of index numbers, no unique index number is acceptable to all, and there is a possibility of manipulation of the base year, price, commodities and quantity quotations.

To conclude we can say that Index numbers are today one of the most widely used statistical devices. They are used to take the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies

5.11 Key Words

- **Index Number:** A ratio of the variable value at the current level to the base level, i.e. the ratio of variable change over a period.
- **Un-weighted Aggregates Index**: Using all the values or data collected for study and allocating same importance to all the values.
- **Un-weighted Average of Relatives Method**: Working out the index number by dividing the present or current level of the variable to its base value, multiplied by 100 and then dividing the summation of percentage values by the number of products to result in average value.
- **Weighted Aggregates Index**: Using all the values considered, but assigning importance or weights to individual ratios.
- **Weighted Average of Relatives Method:** Constructing the index number by allotting weight ages to values of each element in the composite.
- **Consumer Price Index:** Indicates the variations in the prices of a given set of consumer items prepared either at regional level or at National level.
- **Fixed Base Method**: When weight or importance to the variable is allotted based on a given specified fixed period.
- **Price index**: Compares levels of prices from one period (the current) to another (the base) period.
- **Quantity index:** An index comparing the quantity of the variable during a given period by time.

5.12 Self Assessment Test

1. "Index numbers are economic barometers". Explain this statement and mention what precautions should be taken in making use of any published index numbers.

2. Define Index Number and mention its uses.

3. Define index numbers. Distinguish between fixed base and chain base method of constructing index numbers.

4. What are index numbers? What purpose do they serve? Discuss the various problems faced in the construction of index numbers.

5. State and explain Fisher's Ideal Formula for Price Index Number and why is it called Ideal?

6. Write explanatory notes on the following:

   (a) Deflating;
   (b) Splicing
   (c) Base Shifting;

7. Discuss briefly the uses and limitations of index numbers of prices.

8. Compute price index for the following data by (i) simple aggregative method, and (ii) average of price relative method by using arithmetic mean.

Commodities	**A**	**B**	**C**	**D**	**E**	**F**
Price 2003 (Rs.)	20	30	10	25	40	50
Prices 2004 (Rs.)	25	30	15	35	45	55

Ans. (i) 117.14, (ii) 122.9 (by A. M.)

9. Calculate weighted aggregative price index number taking 2001 as base, from the following data:

(Commodity)	**(Quantity consumed)**	**Price per Unit**	
		Base year 2001	**Current year 2004**
Wheat	4 Qtl.	80	200
Rice	1 Qtl.	120	250
Gram	1 Qtl.	100	150
Pulses	2 Qtl.	200	300

Ans. Weighted Index Number = 191.49

10. From the data given in the following table, calculate consumer's price index numbers for the year 2004 taking 2003 as base using (i) simple average, and (ii) weighted average of price relatives:

Items	**Unit**	**Price in Rs.**		**Weight**
		2003	2004	
Wheat	Kg.	0.50	0.75	2
Milk	Liter	0.60	0.75	5
Egg	Dozen	2.00	2.40	4
Sugar	Kg.	1.80	2.10	8
Shoes	Pair	8.00	10.00	1

Ans. (i) 127.34, (ii) 123.3

11. Find out the index number for the year 2004 from the following data using the weighted average of price relatives method:

Commodity	Weight	Price	
		2000	2004
Wheat	4	50	100
Milk	3	30	90
Egg	5	20	10
Sugar	3	60	90
Shoes	5	20	120

Ans.: 270

12. From the following data, calculate Fisher's Ideal Index:

Items	Price per unit (Rs.)		Quantity used	
	2003	2004	2003	2004
A	9.25	15.00	5	5
B	8.00	12.00	10	11
C	4.00	5.00	6	6
D	1.00	1.25	4	8

Ans.: 148.78

13. Given below are two series of index numbers, one based on 1997 and the other on 2000. Splice the new series on 1997 base:

Year	1997	1998	1999	2000	2001	2002	2003	2004
Old Series (A)	100	110	125	150	-	-	-	-
New Series (B)	-	-	-	100	105	120	130	150

Ans.: 100, 110, 125, 150, 157.5, 180, 195, 225

14. From the data given below construct index number of quantities, and of prices for 1970 with 1966 as base using (i) Laspeyre's formula, (ii) Paasche's formula, and (iii) Fisher's Ideal formula.

Commodity	1966		1970	
	Price (Rs.)	Quantity (Units)	Price (Rs.)	Quantity (Units)
A	5.20	100	6	150
B	4.00	80	5	100
C	2.50	60	5	72
D	12.00	30	9	33

Ans. (130.07, 131.02, 130.54) and for price (116.29, 117.14, 116.7)

15. From the following data calculate index numbers of real wages with 1999 as the base:

Year	1996	1997	1998	1999	2000	2001	2002
Average Wages (Rs.)	2400	2640	2860	3000	3420	4000	4200
Consumer Price Index	100	120	130	150	190	200	210

Ans. 120, 110, 110, 100, 90, 100, 100

16. From the data given below, calculate the price index by Fisher's ideal formula and then verify that Fisher's ideal formula satisfies both time reversal test and factor reversal test.

Commodity	*Base year*		*Current year*	
	Price (Rs.)	Quantity ('000 tonnes)	Price (Rs.)	Quantity ('000 tonnes)
A	56	71	50	26
B	32	107	30	83
C	41	62	28	48

Ans. (84.92)

5.13 References

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Unit - 6 : Probability and Probability Distributions

Unit Structure:

6.0 Objectives

After studying this unit, you should be able to understand

- Probability concepts and rules
- Probability distribution Concept
- Binomial distribution and its uses
- Poisson distribution and its uses
- Normal distribution and its uses

6.1 Introduction

Probability has a very old history, it was originated in the games of chance related to gambling. For instance, throwing of dice or coin and drawing cards from a pack. Jerome Cardan (1501~1576), an Italian mathematician was the first man to write a book on the subject "Book on Games of chance" which was published in 1663 after his death. The probability formulae and techniques were developed by Jacob Bernoulli(1654-1705), De Moivre (1667-1754), Thomas Bayes(1702-1761) and Joseph Lagrange(1736-1813). Pierre Simon, Laplace in the nineteenth century unified all these early ideas and compiled the first general theory of probability.

In the beginning, the probability theory was successfully applied at the gambling tables. But after some time, it was applied in the solution of social, political, economic and business problems. In fact, it has become a part of our everyday lives. We face uncertainty in personal and management decisions and use probability theory. Probability constitutes the foundation of statistical theory.

6.2 Approaches

There are mainly three approaches to probability

i. Classical approach

ii. Empirical approach

iii. Axiomatic approach

Few terms can be defined / explained with reference to simple experiments relating to tossing of coins, throwing of a diee or drawing cards from a pack of cards.

Random Experiment

An experiment can be considered as a random experiment if all the possible outcomes are known in advance and none of the outcomes can be predicted with certainty. e.g. throwing a dice, tossing a coin etc.

Trial & Event

When a random experiment is performed, it is called a trial and outcome or combinations of outcomes are termed as events. For example

i. When a coin is tossed repeatedly, the result is not unique. We may get any of the two faces; head or tail. Thus, throwing a coin is a random experiment and getting of a head or tail is an event.

ii. In the similar manner, when a dice is thrown, it is called a random experiment. Getting any of the faces 1, 2, 3, 4, 5 or 6 is an event. Getting an odd no. or an even no., getting no. greater than 3 or lower than five, these are called events.

iii. Similarly, drawing of two balls from an urn containing 'a' red balls and 'b' white balls is a trial and getting of both red balls, or both white balls, or one red and one white ball are events.

Exhaustive Cases

When a random experiment is done, there are some outcomes; the total numbers of possible outcomes are called exhaustive cases for the experiment. For e.g. when a coin is tossed, we can get head (H) or tail (T). Hence exhaustive no. of cases is 2 (i.e. H,T) If two coins are tossed, the various possibilities are HH, HT, TH, TT (number of exhaustive cases are four) where HT means, Head on first coin and Tail on second coin, and TH means, Tail on first coin and Head on second coin and so on.

In case of toss of three coins, number of outcomes is

= (H,T) x (H,T) x(H,T)
= (HH,HT,TH,TT) x (H,T)
= (HHH, HHT, HTH, HTT, THH, THT, TTH, TTT)

No. of possible outcomes is $8 = 2^3$. In general, in a throw of n coins, the exhaustive no. of cases is 2^n.

In a throw of a die, exhaustive number of cases is 6, since we can get any one of the six faces marked 1, 2, 3, 4, 5 and 6. If two dice are thrown, the possible outcomes are

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)

(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(6,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)

i.e. total no. is 36, where (i,j) means number i on the first die and j on the second die, i and j both taking the values from 1 to 6. In the case of throw of two dice, no. of possible outcome = $6^2 = 36$ and in the case of throw of three dice, no. of possible outcome = $6^3 = 216$; in the case of throw of n dice, no. of possible outcome = 6^n

Favourable cases or events

The number of outcomes of a random experiment which result in the happening of an event are named as the cases favourable to the event.

i. When a toss of two coins takes place, the no. of cases favourable to the event 'exactly one head is two (i.e. TH or HT) and for getting 'two heads' is one (i.e. HH)

ii. When a card is drawn from a pack of cards, the no. of cases favourable to the event 'getting a diamond' are 13 and getting 'an ace of spade' is one.

Mutually Exclusive events or cases

Two or more events are considered as mutually exclusive if the happening of any one of them excludes the happening of all others in the same experiment. For example, in toss of a coin, the events 'head' and 'tail' are mutually exclusive because if head comes, we can't get tail and if tail comes we can't get head. Similarly, in the throw of a die, the six faces numbered 1, 2, 3, 4, 5 and 6 are mutually exclusive. Thus, events are said to be mutually exclusive of no two or more of them can happen simultaneously.

Classical/priori Probability

It is the oldest and simplest approach. Under this approach, there is no need to physically perform the experiment. The basic assumption is that the outcomes of a random experiment are equally likely. e.g. in a throw of a dice, occurrence of 1,2,3,4,5,6 are equally likely event.

If a random experiment results in N exhaustive, mutually exclusive and equally likely outcomes out of which m are favourable to the happening of an event X then the probability of occurrence of X i.e. P(X) is given by

$$P(X) = \frac{Favourable \quad cases}{Exhaustive \quad cases}$$

$$= \frac{m}{N}$$

Example 1. A bag containing 10 green and 20 red balls. A ball is drawn at random. What is the probability that it is green.

Sol. Total number of balls in the bag = 10+20 = 30

Number of green balls = 10

Probability of getting a green ball = $\frac{Favourable \quad cases}{Exhaustive \quad cases}$

$$= \frac{10}{30} = \frac{1}{3}$$

Empirical Probability

The classical definition is difficult to apply as soon as we move from the field of coins, cards, dice and other games of chance. It may not explain the actual results in certain cases e.g if a coin is tossed 20 times , we may get 14 heads and 6 tails. The probability of head is thus 0.7 and tail is 0.3. However, if experiment is carried out large number of times, we should expect approximately equal number of heads and tails.

If an experiment is performed **repeatedly** under essentially homogeneous and identical conditions then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials become indefinitely large is known as the probability of happening of the event.

$$P(X) = \lim_{N \to \infty} \frac{m}{N}$$

Axiomatic Approach

The axiomatic Probability theory is an attempt at constructing a theory of probability which is free from inadequacies of both the classical and empirical approaches. It plays an important role in rendering a reasonable amount of comprehensibility and tractability to the understanding of chance phenomenon atleast in the initial stages of any scientific inquiry into their structure and composition where other approaches are less comprehensible and tractable.

6.3 Addition Law

The probability of occurrence of either event A or event B of two mutually exclusive events is equal to the sum of their individual probability.

Mathematically, we can represent as

$$P(A \cup B) = P(A) + P(B)$$

Proof :- If an event A can happen in a_1 ways and B in a_2 ways then

The number of ways in which either event can happen in $a_1 + a_2$ ways.

Total number of possibilities is n.

Then by definition , the probability of either the first or second event happening is

$$\frac{a_1 + a_2}{n} = \frac{a_1}{n} + \frac{a_2}{n}$$

But $\frac{a_1}{n} = P(A)$

And $\frac{a_2}{n} = P(B)$

Hence $P(A \cup B) = P(A) + P(B)$

The theorem can be extended to three or more mutually exclusive events, thus

P(A B C) = P(A) + P (B) + P(C)

Example 2. A deck of 52 cards, one card is drawn. What is the probability that it is either a king or a queen?

Sol. There is four kings and four queens in a pack of 52 cards.

The probability of drawing a card that is king = $\frac{4}{52}$

The probability of drawing a card that is queen = $\frac{4}{52}$

Since the events are mutually exclusive, the probability that the card drawn is either a king or a queen = $\frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

If two events A & B are not **mutually exclusive** (joint events) then the addition law can be stated as follows

The probability of the occurrence of either event A or event B or both is equal to the probability that event A occurs, plus the probability that event B occurs minus the probability that both events occur. I can be shown as

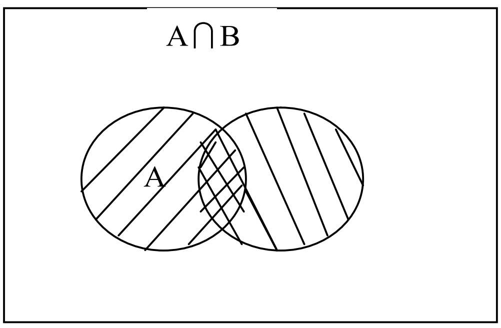
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof.

$P(A \cup B) = n(A \cup B)/n(U)$

Where $n(A \cup B)$ - No. of elements belonging to $(A \cup B)$

And n(U) – total no. of elements in universal set U.



$$P(A \cup B) = \frac{n(A) + n(B) - n(A \cup B)}{n(U)}$$

$$= \frac{n(A)}{n(U)} + \frac{n(B)}{n(U)} - \frac{n(A \cup B)}{n(U)}$$

$$= P(A) + P(B) - P(A \cup B)$$

Example 3. The managing committee of Residents Welfare Association formed a sub-committee of 5 persons to look into electricity problem. Profiles of the 5 persons are

Male age	40
Male age	43
Female age	38
Female age	27
Male age	65

If a chairperson has to be selected from this, what is the probability that he would be either female or over 32 years.

Sol. P (female or over 32) = P(female) + P(over 32) – P(female and over 32)

$$= \frac{2}{5} + \frac{4}{5} - \frac{1}{5} = 1$$

Example 4. What is the probability of picking a card that was a heart or a spade.

Sol. Using the addition rule,

P(heart or spade) = P(heart) + P(spade) – P(heart and spade)

$$= \frac{13}{52} + \frac{13}{52} - \frac{1}{2} = \frac{1}{2}$$

6.4 Multiplication Law

It states that if two events A and B are independent , the probability that they both will occur is equal to the product of their individual probability.

P(A and B) = P(A) X P(B)

It can be extended to three or more independent events.

P(A,B and C) = P(A) X P(B) X P(C)

Proof :- If an event A can happen in n1 ways of which a1 are successful and B in n2 ways of which a2 are successful then

The number of successful happening in both cases is a1x a2 .

Total number of possibilities is n1 x n2.

Then by definition , the probability of the occurrence of both events is

$$\frac{a1 \text{ x } a2}{n1 \text{ x } n2} = \frac{a1}{n} \text{ x } \frac{a2}{n}$$

But $\frac{a1}{n} = P(A)$

and $\frac{a2}{n} = P(B)$

Hence P(A and B) = P(A) x P (B)

Example 5. In order to marry with a girl, a man wants these qualities

White complexion – the probability of getting this is one in fifty.

Etiquettes – the probability is one in hundred.

Dowry – the probability of getting this is one in Twenty.

Calculate the probability of his getting married to such a girl when the possession of these three attributes is independent.

Sol. Probability of a girl with white complexion = $\frac{1}{20}$ = 0.05

Probability of a girl with handsome dowry = $\frac{1}{50}$ = 0.02

Probability of a girl with etiquettes = $\frac{1}{100}$ = 0.01

The probability of simultaneously occurrence of all these qualities = 0.05 x 0.02 x 0.01

= 0.00001

6.5 Conditional Probability

The multiplication theorem described above is not applicable in case of dependent events. Two events A and B are said to be dependent when B can occur only when A is known to have occurred. The probability attached to such an event is called conditional probability and is denoted by P(A/B) i.e. probability of A given that B has occurred.

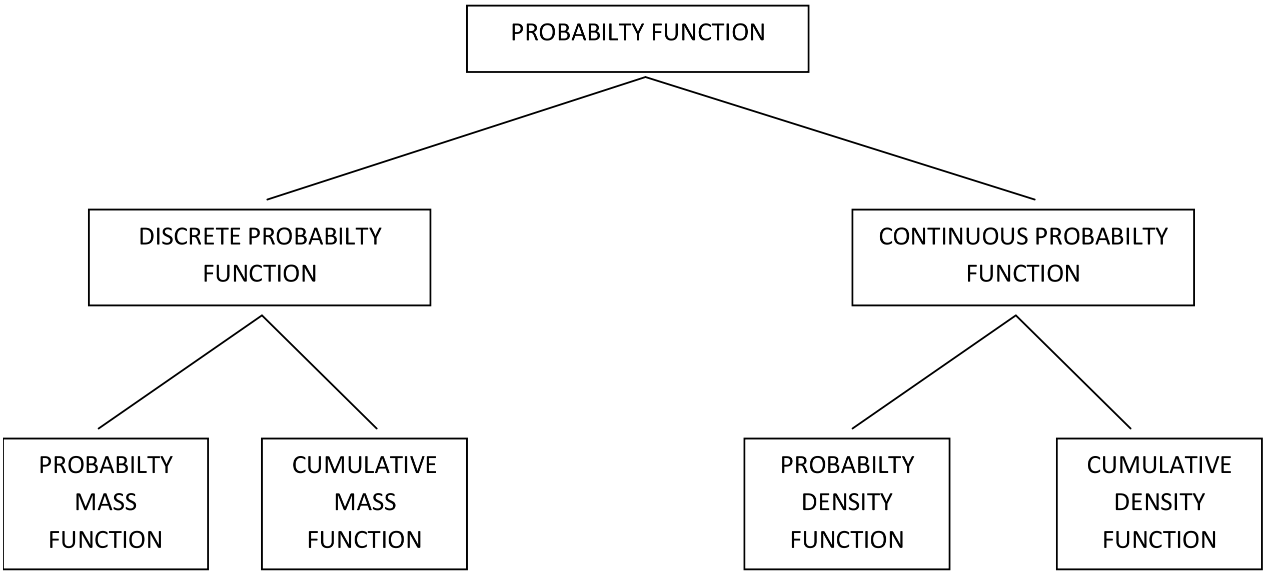
$$
\begin{array}{lll}
 & P(A/B) = P(A \cup B)/P(B); & P(B) \neq 0 \\
\textit{Similary}, & P(B/A) = P(A \cup B)/P(A); & P(A) = 0
\end{array}
$$

Symbolically, we write

$$
\begin{array}{ll}
 & P(A \cap B) = P(A/B) \times P(B) \\
or & P(B \cap A) = P(B/A) \times P(A)
\end{array}
$$

6.6 Probability Function

If the function permits us to compute the probability for any event that is defined in terms of value of the random variable, then the function is called a probability function. Just as there are discrete and continuous random variables, so there are discrete and continuous probability functions. Different types of probability functions are shown:



Discrete Probability Function

A probability function for a discrete random variable is called a discrete probability function since the domain of the function is discrete.

Probability Mass Function

A probability function that specifies the probability that any single value of discrete random variable will occur is called a probability mass function. If f(x) is the probability mass function of the random variable X, then f(x) = P(X=x) has the following properties

(i) $f(x) \geq 0$ for all values of X; and

(ii) $\Sigma f(x) = 1$

Cumulative Mass Function

If X is a discrete random variable with p.m.f f(x), its cumulative mass function (c.m.f.) specifies the probability that an observed value of X will be no greater than X. i.e. if F(x) is a c.m.f. and f(x) is a p.m.f. then $F(x) = P(X \leq x)$.

Continuous Probability Function

A probability function for a continuous random variable is called a continuous probability function since the domain of the function is continuous.

Probability Density function

For a continuous random variable, the corresponding function f(x) is called a probability density function (p.d.f.). Unlike a p.m.f., a p.d.f. doesn't specify probabilities for specific individual value of the random variable.

Cumulative Density function

Corresponding to the cumulative mass function of a discrete random variable, the cumulative density function (c.d.f.) of a continuous random variable specifies the probability that an observed value of X will be no greater than x.

6.7 Bernoulli Distribution

BERNOULLI TRIALS A random variable X which takes two values 0 & 1 with probabilities p and q respectively

It was discovered by James Bernoulli (1654 – 1705). Consider a set of n Bernoulli trials. Following are conditions for Bernoulli trials

a) An experiment is performed for a fixed number of trials.

b) In each trial, there are two possible outcomes of the experiment i.e. success or failure.

c) The probability of a success(p) remains constant from trial to trial. If the probability of success is not the same in each trial, we will not have binomial distribution.

d) The trials are statistically independents i.e. the outcomes of any trial do not affect the outcomes of subsequent trials.

A random variable X is said to be followed binomial probability distribution if it assumes only non negative values and its probability distribution is given by

$$P\ (X = x) = p\ (x) = \{\ {}^{n}c_{x}\ p^{x}\ q^{n-x},\ x = 0, , 2\ \_\ \_\ \_\ \_\ , n$$

A random variable which satisfies conditions of Bernoulli trials, can be represented by Binomial Probability Distribution.

Binomial Distribution is denoted as

$$X \sim B\ (n,p) \text{ where } q = 1 - p$$

n,p are the parameters of binomial distribution and n is the degree of binomial distribution.

Some formulas are

Mean of Binomial Distribution = np

Variance of Binomial distribution = npq

Standard Deviation of Binomial Distribution = $\sqrt{npq}$

Example 6. Find n,p and q when the mean of a binomial distribution is 40 and standard deviation is 6.

Sol. Mean of a binomial distribution = np

Standard deviation of a binomial distribution = $\sqrt{npq}$

$\sqrt{npq}$ = 6 (given) ....(1)

np = 40 (given) ....(2)

$\sqrt{npq}$ = 6 gives

npq = 36; ....(3)

Putting (2) in (3)

$40.\ q = 36, \quad q = 36/40$

$q = 0.9$, putting value in $p = 1 - q$,

gives $p = 0.1$

$np = 40$

$$= \frac{40}{0.1}$$

$$= 400$$

Ans. n=400, p = 0.1, q = 0.9

Example 7. Suppose that the half of the population of town are consumers of rice. One hundred investigators are given the duty to search for the truth. Each investor investigates 10 individuals. How many investigators do you expect to report that three or less of the people interviewed in there sample are consumers of rice?

Sol. Given $n = 10$

$$P = \frac{1}{2}$$

Probability that three people or less consume rice $P(X \leq 3)$

$$= P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3]$$

$$= {}^{10}C_0\, p^0 q^{10} + {}^{10}C_1\, p^1 q^9 + {}^{10}C_2\, p^2 q^8 + {}^{10}C_3\, p^3 q^7$$

$$= (\frac{1}{2})^{10} + 10\,(\frac{1}{2})^{10} + 45\,(\frac{1}{2})^{10} + 120\,(\frac{1}{2})^{10}$$

$$= (\frac{1}{2})^{10}(1 + 10 + 45 + 120)$$

$$= \frac{176}{1024}$$

Therefore the no. of investigators to report that three people or less consume rice is given by NP where N is total number of trials.

$$= \frac{176}{1024} \times 100$$

$$= 17.2 = 17 \text{ (approx.)}$$

Example 8. In an industry, the workers have a 20% chance of suffering from an occupational disease. What is the probability that out of six workers, 4 or more will contract the disease?

Sol. The probability that a worker is suffering from the disease (p) = 20 / 100,

The probability that a worker is not suffering from the disease

$q = 1-p = 1- 1/5 = 4/5$

Probability that 4 people or more; i.e. 4, 5 or 6 will contract disease $P(X \geq 4)$

$$= P[X=4] + P[X=5] + P[X=6]$$

$$= {}^{6}C_{4}\left(\frac{1}{5}\right)^{4}\left(\frac{4}{5}\right)^{2} + {}^{6}C_{5}\left(\frac{1}{5}\right)^{5}\left(\frac{4}{5}\right)^{1} + {}^{6}C_{6}\left(\frac{1}{5}\right)^{6}\left(\frac{4}{5}\right)^{0}$$

$$= 15\left(\frac{1}{5}\right)^{4}\left(\frac{4}{5}\right)^{2} + 6\left(\frac{1}{5}\right)^{5}\left(\frac{4}{5}\right)^{1} + \left(\frac{1}{5}\right)^{6}\left(\frac{4}{5}\right)^{0}$$

$$= 15 \text{ x } \frac{16}{15625} + 6 \text{ x } \frac{4}{15625} + \frac{1}{15625}$$

$$= 0.01696$$

6.8 Poisson Distribution

Poisson distribution was developed by the French mathematician and physicist Simeon Denis Poisson (1781 – 1840). This distribution is frequently used in context of Operation Research. This distribution plays an important role in Inventory control problems, Queuing theory and also in Risk models. Unlike binomial distribution, Poisson distribution can not be deduced on purely theoretical grounds based on the experiment conditions. In fact, it must be based on empirical results of past experiments relating to the problem under study.

Poisson distribution is a limiting case of a binomial distribution under the following conditions

i. n, the no. of trials is indefinitely large i.e. $n \to \infty$ [n is positive integer]

ii. p, the constant probability of success for each trial is indefinitely small i.e. $p \to 0$

iii. np = m is finite where m is a positive real number

Therefore, $\lim\limits_{n \to \infty} b(x; n, p) = \frac{\mathbf{e^{-x}}}{\mathbf{x!}} \mathbf{m^{x}}$

Definition

A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability distribution is given by

$$P(x,m) = P(X=x) = \frac{\mathbf{e^{-x}}}{\mathbf{x!}} \mathbf{m^{x}} \quad ; x = 0, 1, 2, \text{-------------}, \infty$$

Here, m is known as the parameter of the distribution and is equal to np.
Following are conditions for Poisson Distribution

a) It is a discrete probability distribution with

i) Mean = m

ii) Variance = m

b) Total number of terms are “.

c) Probability of success is constant from trial to trial for any given specific interval size.

Example 9. The standard deviation of Poisson variable X is “2, Find the probability that X is strictly positive.

Sol. We know that for Poisson distribution with parameter m,

Variance = m = $(\sqrt{2})^2 = 2$ [Because S.D. = $\sqrt{2}$, given]

Therefore, $P(X = r) = \frac{e^{-m}}{r!} m^r$

$= \frac{e^{-2}}{r!} 2^r$;r = 0,1,2.......

The probability that X is strictly positive is given by

$P(X > 0) = 1 – P (X = 0) = 1 - e^{-2}$

Answer is $1 - e^{-2}$

Example 10. On the basis of past data and experience, it was found that in a plant, there are on an average four accidents per month. Find the probability that in a particular month, there will be less than four accidents. Assuming Poisson distribution ($e^{-4} = 0.0183$)

Sol. We suppose that random variable X indicates the no. of accidents in the plant per month, (m is given as 4 in the usual notation) then by Poisson's probability law

$P(X = r) = \frac{e^{-m}}{r!} m^r$

$= \frac{e^{-4}}{r!} 4^n$

The probability that no. of accidents in a particular month in the plant is less than 4; is given by

$P (X< 4) = P (X = 0) + P (X = 1) + P (X = 2) + P (X = 3)$

$= e^{-4} [1 + 4 + 4^2 / 2! + 4^3 / 3!]$

$= e^{-4} [1 + 4 + 8 + 10.67]$

$= e^{-4} \times 23.67 = 0.0183 \times 23.67 = 0.4332$

Example 11. A manufactured product has 2 defects per unit of product inspected. Using Poisson distribution, calculate the probabilities of finding a product

a) without any defect

b) three defects

c) four defects

Given $e^{-2} = 0.135$

Sol. Average number of defects m = 2

$P(0) = e^{-2} = 0.135$ (Given)

$P(1) = P(0) \times m$

$= 0.135 \times 2$

$= 0.27$

$P(2) = P(1) \times \frac{m}{2}$

$= 0.27 \times \frac{2}{2}$

$= 0.27$

$P(3) = P(2) \times \frac{m}{3}$

$= 0.27 \times \frac{2}{3}$

$= 0.18$

$P(4) = P(3) \times \frac{m}{4}$

$= 0.18 \times \frac{2}{4}$

$= 0.09$

Hence the probability that a product has no defect is 0.135, product has 3 defects is 0.18 and product has 4 defects is 0.09.

6.9 Normal Distribution

The normal distribution was discovered in 1733 by English mathematician De-Moivre who obtained this continuous distribution as a limiting case of the binomial distribution and applied it to problems arising in the game of chance throughout the eighteenth and nineteenth centuries, various efforts were made to establish the normal model as the underlying law ruling all continuous random variables. Thus the name 'normal'. These efforts however failed because of false premises. The normal model has nevertheless become the most important probability model in statistical analysis.

Definition

A random variable X is said to follow a normal distribution with parameters μ (called mean) and variance if its density function is given by the probability law

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

$$\text{or } f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)2}{2\sigma 2}}$$

where $-\infty < x < +\infty$

$-\infty < \mu < +\infty$

$\sigma > 0$

Note

i. A random variable x with mean μ variance σ^2 and following the normal law is expressed by $X \sim N(\mu, \sigma^2)$

ii. If $z = x - \mu / \sigma$

$\varnothing(z) = 1/\sigma\sqrt{2\pi}\ .\ \exp[-z^2/2]$

Mean of $z = E(z) = E[(x-\mu)/\sigma]$

$= E(x) - \mu / \sigma$

$= \mu - \mu / \sigma \quad = 0$

Variance of z = 1

Z is known as standard normal variate

$-\infty < z < +\infty$

6.10 Main Characteristics of Normal Distribution

Normal probability curve

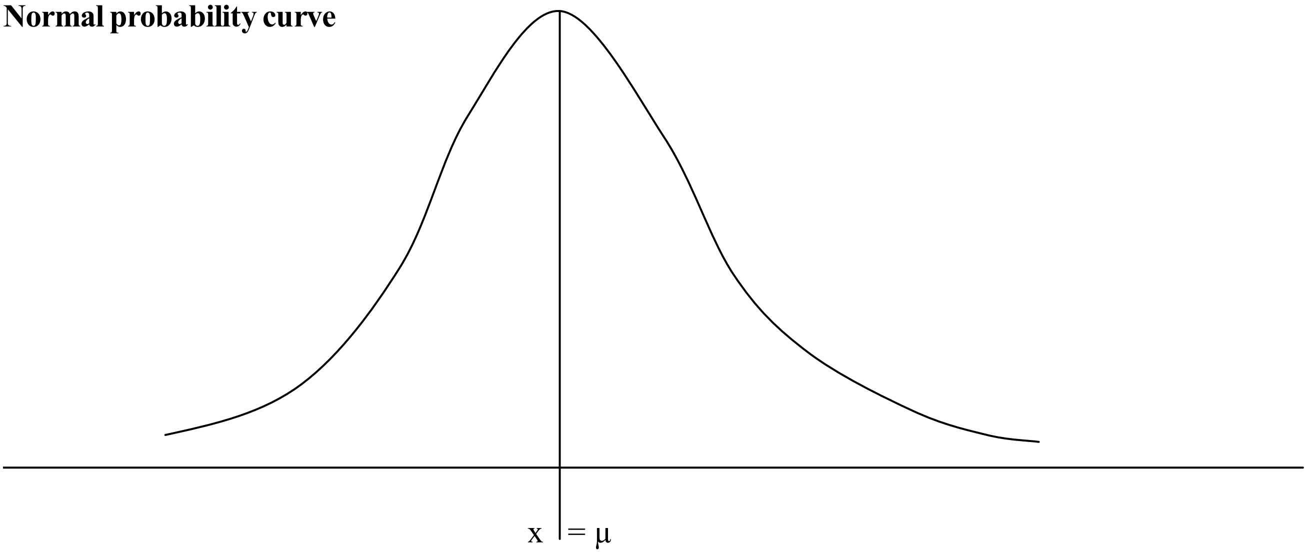


Fig. 6.1

The normal probability curve with mean μ and standard deviation σ is given by the equation

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \qquad -\infty < x < +\infty$$

and has the following characteristics

i. The curve is bell shaped,unimodal and symmetrical about the line x = μ.
ii. Mean, median and mode of the distribution coincide.
iii. Since f(x) being the probability, can never be negative, no portion of the curve lies below the x-axis.

iv. Linear combination of independent normal variates is also a normal variate.

v. X-axis is an asymptote to the curve.

vi. The points of inflexion of the curve are given by [$x = \mu \pm \sigma$, $f(x) = 1/\sigma\sqrt{2\pi} \,.\, e^{-1/2}$]

vii. Area Property

$P(\mu - \sigma < x < \mu + \sigma) = 0.6826$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9544$

$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

(The curve is given below, Fig. 6.2)

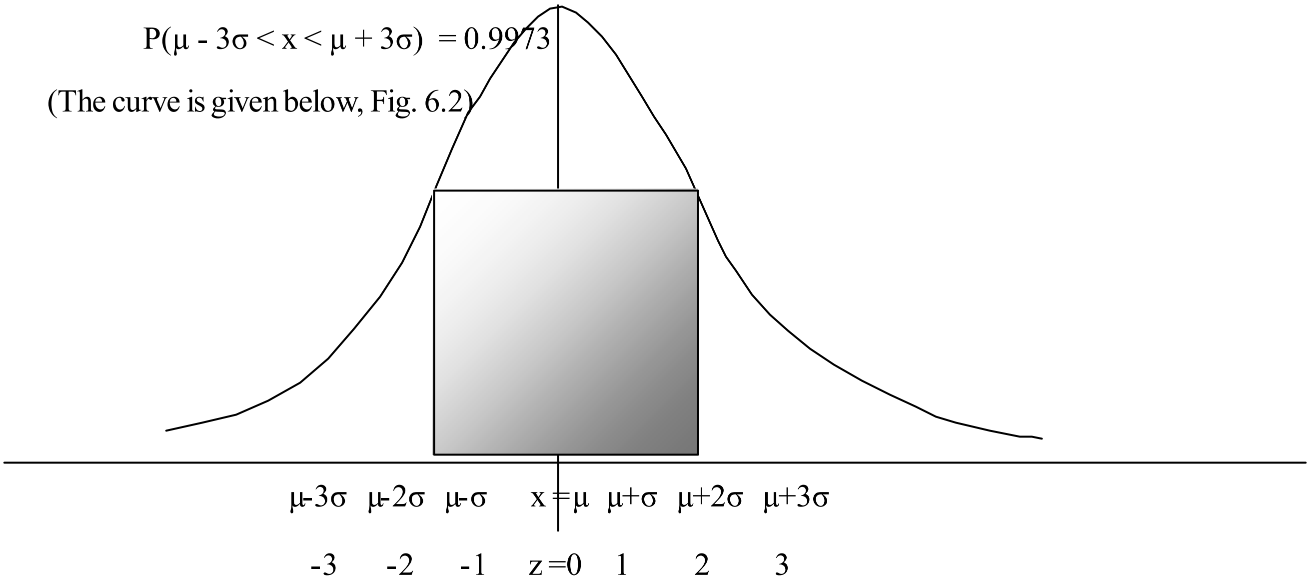


Fig. 6.2

$$P(\mu < x / x_1) = \int_{\mu}^{x_1} f(x)dx = 1/\sigma/2\pi \int_{\mu}^{x_1} fe - \frac{(x-u)^2}{2\sigma^2} dx$$

The probability that random value of x will lie between $x = \mu$ and $x = x_1$ is given by

Put $((x - \mu)/\sigma) = z$

When $x = \mu$, $z = 0$ and

When $x = x_1$, $z = ((x_1 - \mu)/\sigma) = z_1$ (say)

Therefore, $P(\mu < x < x_1) = P(o < z < z_1)$

$$= 1/\sqrt{2\pi} \int_0^{z_1} e^{-\frac{Z^2}{2}} dz = \int_0^{z_1} Ø(z)\, dz$$

Where $Ø(z) = 1/\sqrt{2\pi} \,.\, e^{-\frac{Z^2}{2}}$ is the probability function of standard normal variate. The definite integral

z_1

$\int_{0} \phi(z)\,dz$ is known as normal probability integral and gives the area under standard normal curve between the coordinates at $z = 0$, and $z = z_1$

In particular, the probability that a random value of x lies in the interval $(\mu-\sigma, \mu+\sigma)$ is given by

$$P(\mu-\sigma < x < \mu+\sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$$

$$P(-1 < z < 1) = \int_{-1}^{1} \phi(z)\,dz$$

$$= 2\int_{0}^{1} \phi(z)\,dz$$

$$= 2/\sigma\sqrt{2\pi}\int_{0}^{1} e^{-\frac{1}{2}}\,dz = 2 \times 0.3413$$

$$= 0.6826$$

Similarly,

$$P(\mu-2\sigma < x < \mu+2\sigma)$$

$$= P(-2 < z < 2)$$

$$= 2\int_{0}^{2} \phi(z)\,dz$$

$$= 2 \times 0.4772 = 0.9544$$

and

$$P(\mu-3\sigma < x < \mu+3\sigma) = P(-3 < z < 3) = \int_{-3}^{3} \phi(z)\,dz$$

$$= 2 \times 0.49865 = 0.9973$$

Thus, the probability that a normal variate x lies outside the range $\mu\pm3\sigma$ is given by

$$P[(x-\mu) > 3\sigma] = P(|z| > 3)$$

$$= 1 - P(-3 \le z \le 3)$$

$$= 1 - 0.9973$$

$$= 0.0027$$

6.11 Importance of Normal Distribution

Normal distribution plays a very important role in statistical theory because of the following reasons
ii

i. Most of the distribution occurring in practice e.g. Binomial, Poisson etc. can be approximated by normal distribution.

ii. Even if a variable is not normally distributed it can sometimes be brought to normal form by simple transformation of variable.

i. $P(\mu-3\sigma < x < \mu+3\sigma) = 0.9973$

$P(-3 < z < 3) = 0.9973$

$P(|z| < 3) = 0.9973$

$P(|z| > 3) = 0.0027$

This property of the normal distribution forms the basis of entire large sample theory.

iv. Theory of normal curves can be applied to the graduation of the curves which are not normal.

v. Normal distribution finds large applications in Statistical Quality Control in industry for setting control limits

vi. Many of the distributions of sample statistics (e.g. the distribution of sample mean, sample, variance etc.) tends to normality for large samples and as such they can best be studied with the help of normal curves.

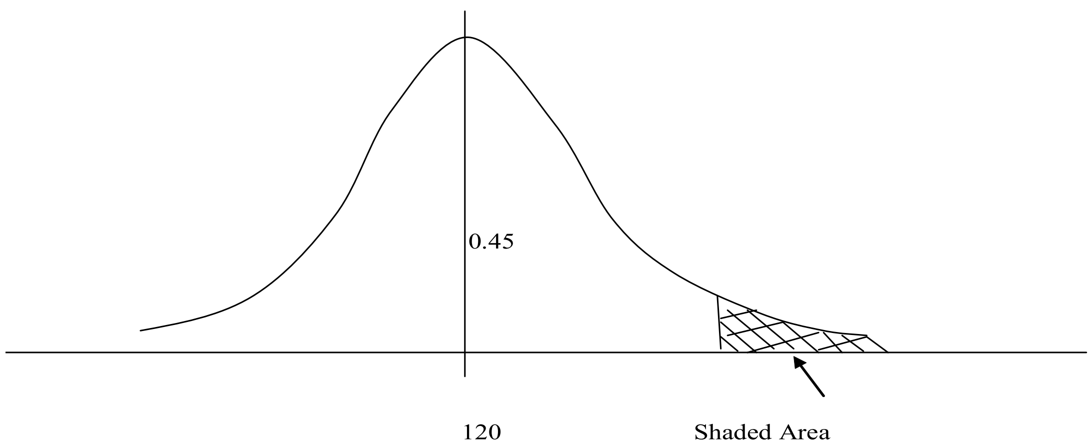
vii. The entire theory of small sample tests is based on the fundamental assumption that the parent populations from which the samples have been drawn follow normal distribution.

Example 12. The annual demand for one type of fertilizer is normally distributed with a mean of 120 tonnes and standard deviation of 16 tonnes. If a wholesale distributor orders only once a year, what quantity should be ordered to ensure that there is only 5% chances of running short of quantity?

Sol. We suppose that the annual demand (in tonnes) be denoted by the random variable X.

Therefore, $z = (X-120) / 16$

The desired area of 0.05 can be shown in the figure below. Since the area between the mean and the given value of X is 0.45, therefore from the table we get this area of 0.45 corresponding to $z = 1.64$



By substituting value of $z = 1.64$ in standardized normal variate, we will get

$1.64 = (X - 120) / 16$

$X = 120 + (1.64)(16) = 146.24$

We can take the order value 146.24 as 147 by rounding if the ordered value is a whole unit.

Example 13. The income distribution of officers of a certain company was normally distributed with a mean income per month Rs. 15,000 and standard deviation Rs. 5,000. If a there were 242 officers, drawing salary above Rs. 18,500, how many officers were there in the company?

[The area under standard normal curve between 0 and 0.7 is 0.2580] **Given**

Sol. We suppose that the income of the officers (in Rs.) be denoted by the random variable X.

Therefore, $X - N(\mu, \sigma)$ where μ = Rs. 15000, and σ = Rs. 5000

Let N be the no. of officers in the company, then the no. of officers in the company drawing salary above Rs. 18500 is

$N.\, P(X > 18500) = 242$ (given) ———————————————————— (1)

When X = 18,500, $Z = (X - \mu) / \sigma = (18500 - 15000) / 5000 = 0.7$

Therefore, $P(X > 18500) = P(Z > 0.7)$

$= 0.5 - P(0 \le Z \le 0.7)$

$= 0.5 - 0.2580$

$= 0.242$

Substituting value of $P(X > 18500)$ in equation (1)

We get from eq. $N.\, P(X > 18500) = 242$

$N.\, 0.242 = 242$

$N = 1000$

Example 14. There are 1000 soldiers in a regiment, it is assumed that the average height of soldiers is 68.22 inches, variance being 10.8 inch^2. How many soldiers in the regiment would you expect to be

i) Over six feet tall ii) less than 5.5 feet.

It is assumed that height is normally distributed.

Sol. We suppose that the height of soldiers (in inches) be denoted by the random variable X. Then, we are given

Mean = $\mu = 68.22$ and

$\sigma = \sqrt{10.8}$

A soldier will be six feet if X is greater than 72 (because X is height in inches).

When X = 72, $Z = (X - \mu) / \sigma = (72 - 68.22) / \sqrt{10.8} = 3.78 / 3.286 = 1.15$

i) The probability that a soldier is over six feet (= 72"), is given by

$P(X > 72) = (Z> 1.15) = 0.5 – P(0 \leq Z \leq 1.15)$

$= 0.5 – 0.3749 = 0.1251$ (From Normal tables)

Hence, in a regiment of 1000 soldiers, the no. of soldiers over six feet is

$1000 \times 0.1251 = 125.1 = 125$ (rounded)

ii) The probability that a soldier is below 5.5'(= 66"), is given by

$P(X < 66) = P(Z < (66 – 68.22) / \sqrt{10.8}) = P(Z < (– 2.22) / 3.286)$

$= P(Z< -0.6756) = P(Z > 0.6756)$

$= 0.5 – P(0 < Z< 0.6756) = 0.5 - 0.2501$ (From Normal tables)

$= 0.2499$ (approx.)

Hence, the no. of soldiers over 5.5' in a regiment of 1000 soldiers is

$1000 \times 0.2499 = 249.9 = 250$ (rounded)

6.12 Summary

This unit provides you a detailed description about probability, various probability laws and probability distribution. Under given conditions, observed frequency distributions can be approximated by well known theoretical distributions. The theoretical distributions indicate as what to expect if a random variable behaves as we assume it does. Probability distributions fall in two categories:

- Discrete
- Continuous

Most important discrete distributions are Binomial and Poisson. Similarly important continuous distribution is Normal distribution.

6.13 Key Words

- **Random Experiment** An experiment can be considered as a random experiment if when conducted rapidly under essentially homogeneous conditions, the result is not unique but may be any one of the various possible outcomes.
- **Trial & event** When a random experiment is performed, it is called a trial and outcome or combinations of outcomes are termed as events.
- **Exhaustive Cases** When a random experiment is done, there are some outcomes; the total numbers of possible outcomes are called exhaustive cases for the experiment.
- **Mutually Exclusive events or cases** Two or more events are considered as mutually exclusive if the happening of any one of them excludes the happening of all others in the same experiment

6.14 Self Assessment Test

1. Explain in detail

    - Binomial Distribution

- Normal Distribution
- Poisson Distribution

2. Explain Probability Function in detail with diagram.

3. Out of 800 families with 4 children each, what percentage would be expected to have

   a) At least one boy

   b) No girls

   c) Two boys and two girls

   d) At most 2 girls

   Assume equal probability for boys and girls.

   (Ans. (a) 93.75% (b) 6.25% (c).37.5% (d) 68.75%)

4. There are 1000 employees, mean of their wages is Rs. 70. Daily wages are distributed normally distributed around the mean and their Standard deviation is Rs. 5. Find out 'Number of workers' whose daily wages will be

   i. Between Rs. 70 and 72

   ii. Between Rs. 69 & 72

   iii. More than Rs. 75

   iv. Also find out the lowest daily wages of the 100 highest paid workers

   (Ans. i. 155, ii. 235, iii. 159, iv. 76.40)

5. A company is manufacturing electric bulbs, if 5% of the electric bulbs manufactured by the company are considered to be defective, use Poisson's distribution to find the probability that in a sample of 100 bulbs

   i. None is defective

   ii. 5 bulbs will be defective ($e^{-5} = 0.007$)

   (Ans. i. 0.007, ii. 0.1823)

6. If on an average 8 ships out of 10 arrive safely at a port, find the mean and standard deviation of the number of ships arriving safely out of a total of 1600 ships.

   (Ans. 1280 ,16)

7. A machine is processing screws and screws are being checked for quality by examining number of defectives in a sample of 6. The following table shows the distribution of 128 samples according to the number of defective items they contained

No. of defectives	0	1	2	3	4	5	6	total
No. of samples	7	6	19	35	30	23	7	128

   i) Fit a binomial distribution and find the expected frequencies if the chance of screw being defective is ½.

ii) Find the mean and standard deviation of the fitted distribution.

8. Ten unbiased coins are tossed concurrently. Find the probability of obtaining

i. Exactly six heads iii At least 8 heads

ii. No head iv At least one head

(Ans. i. 105/512, ii. 1/1024, iii. 7/128, iv. 1023/1024)

9. Find the probability that at most 5 defective bolts will be found in a box of 200 bolts if it is known that 2% of such bolts are expected to be defective. (Given $e^{-4} = 0.0183$)

10. There are 600 business students in the post-graduate department of a university. The probability for any student to need a copy of a particular book from the library on any day is 0.05. How many copies of the book should be kept in library so that the probability may be greater than 0.90 that none of the students needing a copy from the library has to come back disappointed.

(Ans. 37)

6.15 References

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Unit - 7 : Decision Theory

Unit Structure:

7.0 Objectives

After studying this unit, you should be able to understand

- The concept/steps of decision process
- The concept of uncertainty and risk
- Various methods in decision under uncertainty
- Various methods in decision under risk
- The concept and calculation of value of perfect information
- The decision tree and its analysis

7.1 Introduction

Every one of us has to make decisions throughout life. What profession to choose? Where and how much to invest? What to produce and how much to produce? and so on, are some situations where decisions are to be made? Some of the decisions are really difficult to make because of the complexity of the decision situation. This is true in business and industrial problems which are becoming more and more complex. The decision making in business is a very tough task. If someone asks the business executives as to how do they take decision then it will be very difficult question for them to answer because they hardly follow any consistent procedure. Sometimes decision based on cost and profit calculations or at other times based on relative competitiveness. Therefore, we need a decision theory which is defined as a body of methods; helpful to decision maker to select a course of action amongst the alternative plans of actions open to him. Decision Theory is concerned with how to assist organizations in making decisions.

It consists of following steps:

- Decision making Environment
- Objectives of a decision maker
- Alternative plans of action
- Decision payoff

Decision Making Environment

The first step is to know the environment under which a decision is to be taken. Who takes the decision? This is a relevant question. The decision maker may be an individual or a group of individuals. Then they take care of decision situations. These situations can be:

- Situation of certainty
- Situation of Risk
- Situation of Uncertainty

Situation of certainty Certainty situation is one where the outcome of a specified decision can be predetermined with certainty i.e. each action lead to only one outcome. For example if there are three types of machines say X, Y and Z which can take a given job with same level of required accuracy and if the setup time for these machines and operating time per piece on them are known and we are to find out on which machine, an order of given quantity should be processed. Such a situation neither involves competitive nor probabilistic phenomenon. The various important techniques for taking decisions under condition of certainty may be:

a) Techniques used in Transportation Problems

b) Simplex method for LPP

c) Activity Analysis

d) Various mathematical calculations and formulae are available for different applications in business.

Situation of Risk Risk situation is one in which decision maker knows the likelihood that each of the various states of nature will occur i.e. each action will lead to one outcome, each with a known probability. It is a situation where there are many states and the decision maker knows the probability of occurrence of each state. Such decision situations are frequent in business and industry for in many business problems the probabilities of various states are known by determining how frequently they occurred in the past.

Situation of Uncertainty Uncertainty situation is one when the probabilities associated with the state of nature are unknown. Decision making under uncertainty is more difficult than decision making under risk. Many of the major decision problems of business involving situation of uncertainty. For example, if an executive says that the chance that within next six months the project will be completed successfully is 75% then the executive is making a subjective statement and is based on his past managerial experience. Obviously under the situation of uncertainty, there is no one best single criterion for taking a decision.

Decision Stage

Following are the decision stages:

(i) Objectives of a decision Maker

The decisions are made because the decision maker wants to achieve some object. A decision maker will always choose an action that will enable him to achieve his objective. Thus, it is important in the decision process to define explicitly the objectives involved in the decision making process.

(ii) Alternative Plans of Action

If there is only one course of action, there is no decision problem. In a decision process, there are several alternative plans of action. Such alternative plans of action may be limited or unlimited. In the market whether to introduce or not a new product, there are only two alternatives. But for advertisement decision, alternatives would be unlimited. Hence an exhaustive list of all feasible alternative plans of action should be prepared in advance .The problem of decision making is to pick up the best out of these limited or unlimited strategies.

(iii) Decision Payoff

Decision payoff is an indication of the effectiveness of the strategies. In general, effectiveness is measured in terms of money but in many situations it is not possible to give a realistic value of money. Decision payoff's may be fixed decision payoff or can be random variable. In the former case, it is of deterministic nature and in the latter case it is probabilistic in nature. Probability payoff is determined by chance and the strategies chosen.

7.2 One Stage Decision Making Problem

In single stage decision making problems, decisions are taken by considering the pay-offs resulting from various courses of action and outcomes possible.

Example 1. A book shop sells a tax book for Rs. 100. It purchases that book for Rs. 80 per copy. Since tax laws change every year, some copies become outdated and can be disposed of for Rs. 30 each. The annual demand for the book is between 18 and 23 copies according to past experience. Assume that the order can be placed only once during the year, the problem before the seller is to decide how many copies of the book should be purchased for the next year.

Sol. Since the annual demand varies between 18 and 23 copies, there are six possible events:

E_1 : demand for 18 copies
E_2 : demand for 19 copies
E_3 : demand for 20 copies
E_4 : demand for 21 copies
E_5 : demand for 22 copies
E_6 : demand for 23 copies

Also, there are six possible courses of action.

A_1 : buy 18 copies
A_2 : buy 19 copies
A_3 : buy 20 copies
A_4 : buy 21 copies
A_5 : buy 22 copies
A_6 : buy 23 copies

After getting the list of possible actions and events, the next step is to construct the pay-off table.

7.3 Developing Pay-Off Table

A pay-off table shows the economics of the given problem. A pay-off is a conditional value may be conditional cost. Conditional means that it is associated with each course of action given that certain event has occurred. A pay-off table represents the matrix of conditional values associated with all the possible combinations of the acts and the events.

Let D – demand in units for the book

Q – quantity decided to be purchased

Then $P = 20Q$ when $D \geq Q$

$P = 70D - 50Q$ when $D < Q$

The Pay-off table for the above example given below:

Event E_i	Action A_j					
	A_1 : 18	A_1 : 19	A_1 : 20	A_1 : 21	A_1 : 22	A_1 : 23
E_1 : 18	360	310	260	210	160	110
E_2 : 19	360	380	330	280	230	180
E_3 : 20	360	380	400	350	300	250
E_4 : 21	360	380	400	420	370	320
E_5 : 22	360	380	400	420	440	390
E_6 : 23	360	380	400	420	440	460

7.4 Developing Regret Table

The resultant outcomes of the various combinations of the actions and events can be expressed alternatively in terms of regret. It is defined as the amount of pay-off foregone by not adopting the optimal course of action, which would give the highest pay-off for each possible event. For instance, if the demand is 18 copies then the optimal act is to buy 18 copies then regret is zero. Regret table for the problem described in 7.3 given below:

Event E_i	Action A_j					
	A_1 : 18	A_1 : 19	A_1 : 20	A_1 : 21	A_1 : 22	A_1 : 23
E_1 : 18	0	50	100	150	200	250
E_2 : 19	20	0	50	100	150	200
E_3 : 20	40	20	0	50	100	150
E_4 : 21	60	40	20	0	50	100
E_5 : 22	80	60	40	20	0	50
E_6 : 23	100	80	60	40	20	0

7.5 Decision Models

Once the objective, alternative strategies and the decision making environments are known, the next step which a decision maker faces is to select the decision model which can fit into his problem. Some of the models are given below:

- Deterministic Decision Model
- Probabilistic Decision Model
- Competitive Decision Model

Deterministic Decision Model

Deterministic model is related to deterministic situation. Deterministic decision payoffs are the simplest possible payoffs. The objectives and strategies in this model have to be listed and then the payoff for each strategy. If there are two objectives X_1 and $X_{2,}$ the strategies selected are Y_1 and Y_2 and then payoffs are shown below

Objectives → Strategies ↓	X_1	X_2	Total Payoff
Y_1	a_{11}	a_{12}	Σa_{1j}
Y_2	a_{21}	a_{22}	Σa_{2j}

Where a_{ij} refers to payoffs of i^{th} strategy towards j^{th} Objective. In general, with m strategies and n objectives, the decision payoff is as follows:

Objectives → Strategies ↓	X_1	X_2		X_n	Total Payoff
Y_1	a_{11}	a_{12}	...	a_{1n}	Σa_{1j}
Y_2	a_{21}	a_{22}	...	a_{2n}	Σa_{2j}
...	...			...	...
Y_m	a_{m1}	a_{m2}	...	a_{mn}	Σa_{nj}

Optimum strategy would be the one having the largest payoff. Some of the deterministic decision problems can be solved by LPP.

Probabilistic Decision Model

Probability model is related to risk situation. Risk situation is one where there are many states and the decision maker knows the probability of occurrence of each state. Decision payoffs are not fixed but generally a random variable. Payoffs are determined partly by chance and partly by the strategy adopted. Hence, a decision is made in favour of that strategy which has the maximum expected payoff.

Example 2. Suppose a businessman wants to stock commodity A or B. He can stock either but not both. If he stocks A and if it is a success, he can make Rs. 200 but if it is a failure he will lose Rs. 500. If he stocks B and if it a success then he can make 400 but if it is a failure he would lose Rs. 300. Which commodity A or B should he stock? He has the following probability distribution.

Probability Of	With Commodity A's stock	With Commodity B's stock
Success	0.8	0.6
Failure	0.2	0.4

Sol. Payoff matrix given below

	Success	Failure
A	+200	-500
B	+400	-300

Probability matrix is given in problem already. Now the expected payoff matrix is

	Success	Failure	Total Expected Payoff
A	(0.8)(+200)	(0.2)(-500)	160-100 = +60
B	(0.6)(+400)	(0.4)(-300)	240-120 = +120

From the above table, it is clear that business should choose commodity B which has the highest expected payoff = 120.

Decision Under Uncertainty

It is related to the situation of uncertainty. In an uncertainty situation, the probabilities of occurrence of the different events **are not known** and the decision maker has no way of calculating the expected payoff for his strategies. Consequently, there is no single best criterion for selecting a strategy to deal with such a situation but there are different criteria available for selecting a strategy which are given below

a) **Lapalace decision rule**

This rule is based on the assumption that the probabilities of different states of nature for a given strategy are all equal. Considering these equal probabilities the expected payoffs will be calculated and then the strategy with the highest expected payoff is selected. It is also known as the criterion of insufficient reason.

b) **Maximin or Minimax decision rule**

This is the criterion of pessimism. Under this rule the decision maker is pessimistic. As such he selects that strategy which gives largest of the minimum payoffs i.e. the maximum of the minimum. Gains or in case of a loss (Regret) matrix, minimum of the maximum losses. When dealing with the costs, the maximum cost associated with each alternative is considered and the alternative which minimises this maximum cost is chosen. For example the min. profit associated with various strategies is as follows:

S_1: Rs. 520

S_2: Rs. 710

S_3: Rs. 250

S_4: Rs. 290

S_5: Rs. 310

S_6: Rs. 460

Since the maximum of these is Rs. 710, the strategy S_2 is selected corresponding to the maximin principle.

(c) Maximax or Minimin decision rule

This is the criterion of optimism. Under this rule the decision maker is optimistic. As such he selects that strategy which gives him the best possible payoff or best of bests. If it is a profit matrix, the decision maker selects a strategy which gives him the highest of the maximum payoffs i.e. maximum of the maximum or in case of loss (Regret) matrix, the minimum of minimum losses. For example maximum payoff associated with the different strategies is given below:

S_1: Rs. 510

S_2: Rs. 740

S_3: Rs. 210

S_4: Rs. 290

S_5: Rs. 370

S_6: Rs. 980

The highest profit is Rs. 980, strategy S_6 of ordering 40 copies of the magazines is the decision according to the maximax principle.

(d) HURICZ decision rule

According to this rule, it is assumed that the decision maker is α pessimist and $(1-\alpha)$ optimist. A linear combination of the values pessimist and optimist are calculated for each action. The one with maximum expected profit is chosen. Under these criteria the final choice is governed by the level of optimism/pessimism of the decision maker i.e. the choice may differ for different decision maker according to their level of optimism/pessimism.

(e) Savage decision rule

This rule is based on general insurance against risk. Under this rule, one selects the strategy which causes minimum of the maximum possible losses. In this, payoff matrix is converted into a loss or regret matrix. In each cell, we enter the difference between what the decision maker would have done if he had known which outcome would occur and the choice represented by the cell. Once the regret matrix is formed, the minimax criterion can be applied to it to select the best course of action.

Example 3. A newspaper boy purchases n number of copies at the rate of Rs. 8/- per copy and sells at the rate of Rs. 10/- per copy. Any paper remaining unsold at the end of the day is disposed off as waste at Re. 1/- per copy. From past experience, it is known that the demand for such copies has always varied between 21-25. If all the copies are sold and there is a demand for more copies, the newspaper boy does not get another chance to go and purchase additional copies to be sold. This problem is to decide the number of copies that he should stock in the morning so as to maximize his profit.

Sol. Step 1: Set up the conditional Payoff Matrix as given below:

Possible Actions

State of Nature (Demand of copies in the market)	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
D_1 (21)	42	35	28	21	14
D_2 (22)	42	44	37	30	23
D_3 (23)	42	44	46	39	32
D_4 (24)	42	44	46	48	41
D_5 (25)	42	44	46	48	50

Table 1

The cell entries are calculated as follows:

e.g. for cell(2,3). 23 copies stocked, 22 sold (demand being 22)

22 copies sold @ 10/-	=	220
1 copy sold @ 1/-	=	1
Total Revenue	=	221
Cost of 23 copies @ Rs. 8/-	=	184
Total conditional profit	=	37

Similarly calculations are done for all other cells.

Answer according to

a) Laplace Decision Rule:

Calculate average for each action for Table 1. This is known as expected value for each action.

	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
Average:	**42**	**42.2**	**40.6**	**37.2**	**32**

(Expected Value)

Since expected value for Action A_2 **(22)** is maximum, the decision is to stock 22 copies in the morning with an average profit of Rs. 42.2/-.

b) Maxmin decision rule (Also known as Pessimistic Approach)

The decision maker identifies the minimum profit that he can get for each of his decision choice (actions) and chooses the one which is maximum among these. e.g. with reference to Table 1

	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
Min. profit:	**42**	**35**	**28**	**21**	**14**

Choose maximum from among these.

42 being the maximum, the decision is to stock 21 copies in the morning with an average profit of Rs. 42/-

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a) Maximax decision rule (Also known as Optimistic Approach)

The decision maker identifies the maximum obtainable profit for each of his decision choice (actions) and chooses the one which is maximum among these.

	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
Max. profit:	**42**	**44**	**46**	**48**	**50**

Choose maximum from among these.

Maximum of the maximum being 50; the decision is A_5 i.e. stock 25 in the morning for an average profit of Rs. 50/- .

b) HURWICZ decision rule:

The calculations are as follows

Previously the maxmin and maxmax; pessimistic and optimistic calculations have been done.

Now suppose the level of pessimism is 0.3 and

Therefore the level of optimism is $(1-\alpha) = (1- 0.3) = 0.7$

	V_P	V_0	V_H	
A_1 (21)	42	42	42	(42*0.3 + 42*0.7) = 42
A_2 (22)	35	44	41.3	(35*0.3 + 44*0.7) = 41.3
A_3 (23)	28	46	40.6	(28*0.3 + 46*0.7) = 40.6
A_4 (24)	21	48	39.9	(21*0.3 + 48*0.7) = 39.9
A_5 (25)	14	50	39.2	(14*0.3 + 50*0.7) = 39.2

V_p – Value for action under pessimist criteria

V_o – Value for action under optimist criteria

V_H – Value for action under HURWICZ criteria

According to the above calculations, the HURWICZ value V_H (for α=0.3 and $(1-\alpha) = 0.7$) is 42 i.e. maximum for A_1.

Therefore according to this criterion the decision is to stock 21 copies in the morning with expected profit of rs. 42/- .

c) Minimax Regret criteria

In this case first of all, using the conditional payoff matrix, a regret table is generated.

For each state of nature (rows in this case) all values of the row are subtracted from the maximum value of the row. This is done for all rows. Accordingly the regret matrix is obtained as follows.

Possible Actions

State of Nature (Demand of copies in the market)	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
D_1 (21)	42	35	28	21	14
D_2 (22)	42	44	37	30	23
D_3 (23)	42	44	46	39	32
D_4 (24)	42	44	46	48	41
D_5 (25)	42	44	46	48	50

Possible Actions

State of Nature (Demand of copies in the market)	A_1 (21)	A_2 (22)	A_3 (23)	A_4 (24)	A_5 (25)
D_1 (21)	0	7	14	21	28
D_2 (22)	2	0	7	14	21
D_3 (23)	4	2	0	7	14
D_4 (24)	6	4	2	0	7
D_5 (25)	8	6	4	2	0
maximum regret for each action	8	7	14	21	28
	choose minimum regret				

The minimum being 7, the decision is: A_2 i.e. stock 22 copies in the morning with the average regret value = 7.

This decision is same as was obtained under max min profit criteria. The decision resultant from maximum profit and min max regret will always be same.

Decision under Risk

Risk situation is slightly better than uncertainty in the sense that in risk situation probability of occurrence of States of Nature are known. The decision maker takes a well considered and calculated decision and risk.

Example 4: Let us consider the same example of newspaper boy with additional information that now probabilities of States of Nature (demand) is known and stated. The resultant conditional pay off matrix is as follows:

Decision Maker
Possible Actions:
(No. of copies he can stock)

States of Nature (Demand of Copies in the Market)	P	A_1 21	A_2 22	A_3 23	A_4 24	A_5 25
D_1 (21)	0.1	42	35	28	21	14
D_2(22)	0.15	42	44	37	30	23
D_3(23)	0.45	42	44	46	39	32
D_4(24)	0.2	42	44	46	48	41
D_5 (25)	0.1	42	44	46	48	50
E (A_i)		**42**	**43.1**	**42.85**	**38.55**	**32.45**

Criteria used in decision making under uncertainty

(i) Criteria of Maximum likelihood.

(ii) Criteria of maximum expectation

(iii) Criteria of minimum expected regret / opportunity loss.

(i) Criteria of Maximum likelihood:

According to these criteria the decision maker chooses action for the event for which probability of happening in maximum. For example: in the pay off matrix given Example 2; 45 is the maximum profitability for occurrence of demand of 23 newspapers.

For demand of 23 news papers, his maximum profit is Rs 46/- against action A_3 = Stock 23 papers. Therefore he will stock 23 news papers in the morning.

(ii) Criteria of Maximum Expectation:

According to these criteria; the decision maker will choose that Action (Ai) for which it expected value is maximum. For action Ai the expected value E (Ai) is calculated as follows:

$$A_3 = 0.1 \times 28 + 0.15 \times 37 + 0.45 \times 46 + 0.2 \times 46 + 0.1 \times 46$$
$$= 42.85$$

Similarly, the calculations for all Actions (columns) are completed. The calculated values are shown in last row in table against.

Actions		**A_1**	**A_2**	**A_3**	**A_4**		**A_5**
E (A_1)		**42**	**43.1**	**42.85**	**38.55**		**32.55**

The maximum expected value among E (Ai) being 43% for action A_2; the decision will be the stock 22 copies for an expected profit of 43.1 Rupees.

(iii) Criteria of Minimum Expected Regret

Consider the regret table generated in earlier case (Decision under uncertainty with minimum regret); along with the probabilities considered in the previous example:

(Regret Table)

Possible Actions: No. of copies he can stock

States of Nature (Demand of Copies in the Market)	**P**	**A_1 21**	**A_2 22**	**A_3 23**	**A_4 24**	**A_5 25**
D_1 (21)	0.1	0	7	14	21	28
D_2(22)	0.15	2	0	7	14	21
D_3(23)	0.45	4	2	0	7	14
D_4(24)	0.2	6	4	2	0	7
D_5 (25)	0.1	8	6	4	2	0
Expected Regret For Each Action		**4.1**	**3**	**3.25**	**7.55**	**13.65**

↑ Minimum **Regret**

Therefore under these criteria, the choice is stock 22 copies with minimum expected regret of 3. The decision under these criteria will always be the same as under criteria of maximum expectation, discussed earlier.

Value of Perfect Information:

In the above discussion, under criteria of maximum expectation, the decision is

A_2 – to stock 22 copies with an expected value of 43.1 Rupees

This means the decision maker can realize an average profit of Rs. 43.1 / day, if he uses the strategy over large number of 6 days.

Now, suppose there is a market research expert who claims that he can exactly predict demand for the day. So that the decision maker can make the best choice every day depending upon that days prediction. So, if he is able to tell exactly each day's demand; he is providing PERFECT INFORMATION to the decision maker for the problem situation. Therefore on the basis of this PERFECT INFORMATION the decision maker is able to take the best decision each day.

States of Nature (Demand of Copies in the Market)	P	A_1 21	A_2 22	A_3 23	A_4 24	A_5 25
D_1 (21)	0.1	42	35	28	21	14
D_2(22)	0.15	42	44	37	30	23
D_3(23)	0.45	42	44	46	39	32
D_4(24)	0.2	42	44	46	48	41
D_5 (25)	0.1	42	44	46	48	50
E (A_i)		**42**	**43.1**	**42.85**	**38.55**	**32.45**

As per above table:

Expected value under Perfect Information:

$$EVPI = 42 \times 0.1 + 44 \times 0.15 + 46 \times 0.45 + 48 \times 0.2 + 50 \times 0.1$$
$$= 46.1$$

Therefore, the value of perfect information

VPI = Expected value under perfect information – Maximum expected value
(As calculated under criteria of maximum expectation)

$$= 46.1 - 43.1$$
$$= 3.0$$

Rs 3/- is the value of perfect information. It is not advisable to pay to a consultant or marketing researcher or advisor a fee is more than its value; in this case it is Rs 3/-.

Thus, it can be said that value of information is the ADDITIONAL amount (over and above what could be earned otherwise) that can be earned using the information.

Decision Tree

In some situations the decision problem can be represented graphically known as Decision Tree of Decision Flow Diagram. This type of representation is more useful, where sequences of decisions are required to be taken one after the other. This is termed as multistage decision problem. That means some times a decision situation can be graphically represented as one stage, two stage or multistage problem; and accordingly analyzed to arrive at a decision.

Example 5: A Company is currently working with a process, which, after paying for materials, labour and so on brings a profit of Rs. 12,000. The company has the following alternatives:

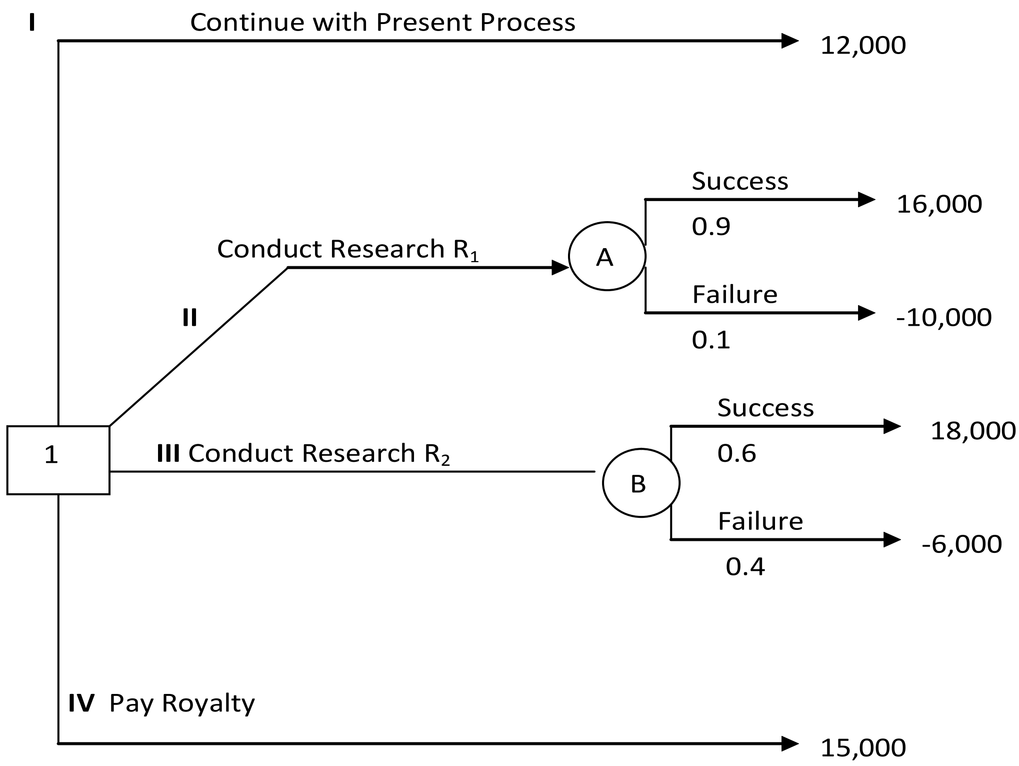
(i) The Company can conduct research R_1 which is expected to cost Rs 10,000 and having 90% probability of success. IF successful, the gross income will be Rs 26,000.

(ii) The company can conduct research R_2, expected to cost Rs 6,000 and having a probability of 60% success. It successful, the gross income will be Rs 24,000.

(iii) The company can pay Rs 5,000 as royalty of a new process which will bring a gross income of Rs 20,000.

Because of limited resources, only one of the two types of research can be carried out at a time. Draw the decision tree and bring out the optimal strategy for the company.

Decision tree corresponding to the above example is shown below:



Analysis:

The situation has four alternatives I, II, III, IV as shown in the above figure. This is a one stage problem.

The decision point is shown as a box. $\boxed{1}$ Small ◯ show the possible outcome at those points; A and B in this case:

The process of Analysis is shown in table below:

Alternative	**Net Expected Outcome**
I) Continue with Present Process	Given = 12,000
II) Conduct Research R_1	$16,000 \times 0.9 - 10,000 \times 0.1 = 13,400$
III) Conduct Research R_2	$18,000 \times 0.6 - 6,000 \times 0.4 = 8,400$
IV) Pay Royalty for new process	$20,000 - 5,000 = 15,000$

Since the expected outcome 15,000 is highest among all the four alternatives, it is recommended that alternative IV i.e. "Pay royalty of 5,000/- for a new process to earn gross income of Rs. 20,000 be selected.

Example 6. Aldico Ltd. has installed a machine costing Rs. 4 lacs and deciding an appropriate number of a certain spare parts required. Cost of spare part each Rs. 4000 but are available only if they are ordered now. If machine fails and no spare parts are available, the cost to the company of mending the plant would be Rs. 18000. Estimated life of the plant is of 8 years and the probability distribution of failures during this time is as follows

No. of failures during 8 yearly period	Probability
0	0.1
1	0.2
2	0.3
3	0.2
4	0.1
5	0.1
6	0

Determine the following

i) The optimal number of units of spare parts on the basis of

(a) Minimax principle
(b) Minimin principle
(c) Laplace principle

ii) Expected number of failures in the 8 yearly periods.
iii) Regret table
iv) EVPI.

F - number of failures
S - number of spares
C – total cost
Cost function can be stated as follows

$C = 4000S \quad$ when $F \leq S$
$\quad = 4000S + 18000\,(F - S) \quad$ when $F > S$

Sol. Cost matrix shown below

No. of Failures	Probability	No. of Spares					
		$S_1 : 0$	$S_2 : 1$	$S_3 : 2$	$S_4 : 3$	$S_5 : 4$	$S_6 : 5$
$F_1 : 0$	0.1	0	4	8	12	16	20
$F_2 : 1$	0.2	18	4	8	12	16	20
$F_3 : 2$	0.3	36	22	8	12	16	20
$F_4 : 3$	0.2	54	40	26	12	16	20
$F_5 : 4$	0.1	72	58	44	30	16	20
$F_6 : 5$	0.1	90	76	62	48	34	20
Column Minima		**0**	**4**	**8**	**12**	**16**	**20**
Column Maxima		**90**	**76**	**62**	**48**	**34**	**20**
Simple Average Cost		**45**	**34**	**26**	**21**	**19**	**20**
Expected Cost		**41.4**	**29.2**	**20.6**	**17.4**	**17.8**	**20**

i) No. of units according to

a) Minimax : The maximum values in each of the columns are shown by the column maxima row. The minimum of these is 20. Hence the decision would be to buy 5 spare parts.

b) Minimin : The minimum values in each of the columns are shown by the column minima row. The minimum of these is 0. Hence the decision would be to buy nothing.

c) Laplace Principle: According to this, decision is taken on the basis of the simple average cost values. Since the simple average cost is minimum for S_5. Hence the decision would be to buy 4 spare parts.

ii) Expected number of failures in the 8 year period.

$$E(F) = \sum_{i=1}^{6} p_i E_i$$

Thus E(F) = 0.1*0 + 0.2*1 + 0.3*2 + 0.2*3 + 0.1*4 + 0.1*5 = 2.3

iii) Regret Table is given below

No. of Failures	Probability	No. of Spares					
		$S_1 : 0$	$S_2 : 1$	$S_3 : 2$	$S_4 : 3$	$S_5 : 4$	$S_6 : 5$
$F_1 : 0$	0.1	0	4	8	12	16	20
$F_2 : 1$	0.2	14	0	4	8	12	16
$F_3 : 2$	0.3	28	14	0	4	8	12
$F_4 : 3$	0.2	42	28	14	0	4	8
$F_5 : 4$	0.1	56	42	28	14	0	4
$F_6 : 5$	0.1	70	56	42	28	14	0
Expected Regret		**32.2**	**20**	**11.4**	**8.2**	**8.6**	**10.8**

Expected regret for S_4 is the least. Hence the decision would be to buy 3 spare parts.

iv) EVPI (Expected value of perfect Information)

EVPI = Expected cost with optimal policy – Expected cost with perfect information

Expected cost with optimal policy is 17.4 thousand rupees (Given in problem)

Event	Cost	Prob.	Prob. X Cost
$F_1 : 0$	0	0.1	0.0
$F_2 : 1$	4	0.2	0.8
$F_3 : 2$	8	0.3	2.4
$F_4 : 3$	12	0.2	2.4
$F_5 : 4$	16	0.1	1.6
$F_6 : 5$	20	0.1	2.0
Expected Cost			9.2

EVPI = 17.4 – 9.2 = 8.2 thousand rupees.

7.6 Bayesian Decision Rule: Posterior Analysis

In this approach, the optimal strategy is chosen using the expected value criteria while the expected pay-offs are calculated by using posterior probabilities. In this rule, prior information of the decision maker is revised on the basis of some additional information i.e. the using this information, prior probabilities are converted into the posterior probabilities.

Example 7: A construction company has recently acquired a piece of land and plans to construct a shopping Mall. The company has to decide the size of the complex. There are three options:

a) A small – sized complex with 40 condominiums and a multiplex

b) A medium – sized complex with 60 condominiums and a multiplex

c) A large – sized complex with 100 condominiums and a multiplex

The company feels that the overall demand for the condominiums built would be either high or low. The returns from the project will depend on size of complex and what the level of demand eventually turns out to be. The pay-offs expected under various event-action combinations, together with estimated probabilities of the demand are given below

Table 1

Event	Probability	Actions A_j		
		Small Complex	Medium Complex	Large Complex
High demand	0.4	1800	2200	4200
Low demand	0.6	1000	600	-1200

Company has the choice of engaging a marketing research company to conduct a survey so that it can take a more informed decision. Outcome of research as follows:

A **favourable** report indicating high demand

An **unfavourable** report indicating low demand

The past record of the research firm has led to the following estimates of the relevant probabilities

Table 2

Event	Marketing Research Report	
	Favourable	Unfavourable
High demand	0.9	0.1
Low demand	0.2	0.8

The fee of marketing research Firm is Rs. 30000 for this study. How should the construction company proceed?

Sol. The company has two options to decide on the size of the complex.

a) Prior analysis

b) Posterior analysis

Prior analysis: The pay-off table is reproduced from table 1 and is shown below

Event	Probability	Actions A_j		
		Small Complex	Medium Complex	Large Complex
High demand	0.4	1800	2200	4200
Low demand	0.6	1000	600	-1200
Expected Pay-off		1320	1240	960

Here the expected pay-off for Small Complex is maximum i.e. optimal course of action.

Event	Probability	Pay-off	Pay-off X Probability
E_1 : High demand	0.4	4200	1680
E_2 : Low demand	0.6	1000	600
EPPI			2280

$$EVPI = EPPI - EP = 2280 - 1320 = 960$$

Posterior analysis: Let us consider how posterior probabilities may be calculated and used. It may be noted that the probabilities given in past records are in fact conditional probabilities.

$P(I_1 / E_1)$ - probability that research firm gives a favourable report given state of nature eventually turns out to be higher demand.

$P(I_1 / E_2)$ - probability that research firm gives a favourable report given state of nature eventually turns out to be lower demand.

$P(I_2 / E_1)$ - probability that research firm gives an unfavourable report given state of nature eventually turns out to be higher demand.

$P(I_2 / E_2)$ - probability that research firm gives an unfavourable report given state of nature eventually turns out to be lower demand.

Total probability that the research report will be favourable

$$P(I_1) = P(E_1 \cap I_1) + P(E_2 \cap I_1)$$

$= P(E_1) X P(I_1 / E_1) + P(E_2) X P(I_1 / E_2)$

$= 0.4*0.9 + 0.6*0.20$

$= 0.36 + 0.12$

$= 0.48$

Total probability that the research report will be unfavourable

$$P(I_1) = P(E_1 \cap I_2) + P(E_2 \cap I_2)$$

$= P(E_1) X P(I_2 / E_1) + P(E_2) X P(I_2 / E_2)$

$= 0.4*0.1 + 0.6*0.80$

$= 0.04 + 0.48$

$= 0.52$

These posterior probabilities can be used to determine optimal course of action under each of these situations.

a) When a favourable report is given

The posterior probabilities for the events E_1 and E_2 when a favourable report is given by marketing research firm, are shown below

Event E_i	Prior probability $P(E_i)$	Conditional Probability $P(I_1 / E_i)$	Joint Probability $P(E_i \cap I_1)$	Posterior Probability $P(E_i / I_1)$
E_1	0.4	0.90	0.36	0.36/0.48=0.75
E_2	0.6	0.20	0.12	0.12/0.48=0.25

The posterior probabilities of the event, the pay-off matrix and the expected values using these probabilities are shown below

Event E_i	Probability p_i	Action_j		
		Small Complex	Medium Complex	Large Complex
E_1 : High demand	0.75	1800	2200	4200
E_2 : Low demand	0.25	1000	600	-1200
Expected Pay-off		1600	1800	2850

Since the expected pay-off for the action Large Complex is highest. It is concluded that given a favourable research report is obtained, the best course of action is large sized shopping complex.

b) When an unfavourable report is given

The posterior probabilities for the events E_1 and E_2 when an unfavourable report is given by marketing research firm, are shown below

Event E_i	Prior probability $P(E_i)$	Conditional Probability $P(I_2 / E_i)$	Joint Probability $P(E_i \cap I_2)$	Posterior Probability $P(E_i / I_2)$
E_1	0.4	0.10	0.04	0.04/0.52=0.0769
E_2	0.6	0.80	0.48	0.48/0.52=0.9231

The posterior probabilities of the event, the pay-off matrix and the expected values using these probabilities are shown below

Event E_i	Probability p_i	Action_j		
		Small Complex	Medium Complex	Large Complex
E_1 : High demand	0.0769	1800	2200	4200
E_2 : Low demand	0.9231	1000	600	-1200
Expected Pay-off		1061.52	723.04	-784.74

Since the expected pay-off for the action Small Complex is highest. It is concluded that given a favourable research report is obtained, the best course of action is small sized shopping complex.

Thus when a favourable market research report is obtained, the optimal course of action is Large Complex and expected pay-off is of Rs. 2850.

When an unfavourable market research report is obtained, the optimal course of action is Small Complex and expected pay-off is of Rs. 1061.52.

But these decisions are conditional in the sense that either of them can be taken only when the nature of report is known. Expected pay-off value is shown below

Indicator	Probability	Conditional Pay-off of the Best Action	Expected Value
Favourable Report	0.48	2850	1368
Unfavourable Report	0.52	1061.52	552
Expected Pay-off			1920

Expected pay-off of the optimal decision is Rs. 1920 if it is based on the market research information as against Rs. 1320 without such information.

Expected value of Sample Information (EVSI)

EVSI = Expected pay-off with sample information – Expected pay-off without sample Information

= 1920 - 1320

= 600 thousand

The company can pay an amount of Rs. 600000 for research report. But firm has asked a fee of Rs. 300000. It is worth engaging it.

7.7 Limitations

1. The prior knowledge of all the possible outcomes is necessary to be able to use the techniques / methods used in decision theory.
2. For construction of payoff table / regret table the monetary value of combination of actions & possible states of nature; is to be known before hand.
3. The decisions are probabilistic in nature, therefore the outcome of decisions not definite.

7.8 Summary

In this section you have learned how to develop payoff/ regret tables. Various techniques/methods for taking decisions under uncertainty using the payoff/regret table have been discussed. These decisions are under the assumption of single stage decision situations. The techniques available for decision under risk (i.e. when the probabilities of the occurrence of events are known). The concept of value of information and value of perfect information is also discussed .Finally the construction and analysis of decision tree is discussed. Application of decision tree is comparatively more useful for multistage decision situations.

7.9 Key Words

- **Uncertainty:** When the probabilities w. r. t. to happening of events (States of Nature) are not known.
- **Risk :** When the probabilities w. r. t. to happening of events (States of Nature) are known.
- **Pay of Matrix:** A Matrix showing profit / loss for all combinations of Action and Possible events (States of Nature).

- **Regret Matrix:** A Matrix (table) matrix showing regret /opportunity loss for not having taken the best decision.
- **Decision Tree:** A graphical representation (in the form of branches) various decision points along with the alternatives; and possible outcome with their probability events of happening. It also shows in monitory term the profit / loss for each combination alternative event happening.

7.10 Self Assessment Test

1. Briefly explain the different decision rules adopted in decision making under condition of uncertainty.

2. Discuss the main features of the decision theory. Also state its limitations.

3. Write short note on the following

   a) Deterministic decision Model

   b) Probabilistic decision Model

   c) Decision Payoff

4. Define EVPI. How it is calculated?

5. Explain the posterior analysis of decision making.

6. From the given matrix, elements of which indicate profits, obtain the decisions using the following principles of decision making

   a) Maximax

   b) Maximin

   c) Laplace

	a_1	a_2	a_3	a_4	a_5
S_1	26	22	13	22	18
S_2	26	22	34	30	20
S_3	18	22	18	18	20
S_4	22	22	18	18	18

7. The Jeans manufacturer must decide whether to build a large factory or a small factory in a particular location. The profit per pair of jeans manufactured is estimated as Rs. 10. A small factory will have an amortised annual cost of rs. 200000 with a production capacity of 50000 jeans per year. A large factory will have an amortised annual cost of Rs. 400000 with a production capacity of 100000 jeans per year. Four levels of manufacturing demand are considered equally likely namely 10000, 20000, 50000 and 100000 pair of jeans per year.

   a) Set up a pay-off table for building a small factory and a large factory for manufacturing designer jeans.

   b) Explain the meaning of EVPI.

   c) Which would you choose to build, a small or large factory? Why?

7.11 References

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Unit - 8: Operations Research

Unit Structure:

8.0 Objectives

After studying this unit, you should be able to understand

- Meaning of Operations Research
- Characteristics of Operations Research
- Application areas of Operations Research
- Operations Research Models, their advantage and disadvantages
- How to construct an OR model
- Various types of mathematical models
- Operations Research scope
- Operations research development in India
- Impact of Computers in Operations Research

8.1 Introduction

Operation Research (OR) came into existence during the World-War II. In England, Military management called upon a team of scientists to study the strategic and tactical problems at that time related to land and air defence of the country. Their mission was to formulate specific proposals and plans for aiding the Military commands to arrive at the decisions on optimal utilization of military resources and to implement the decisions effectively also.

OR can be associated with "AN ART OF WINNING THE WAR WITHOUT ACTUALLY FIGHTING IT." OR teams were not actually engaged in military operations and in fighting the war. But they were advisors and instrumental in winning the war to the extent that systematic approaches involved in OR provides an intellectual support to strategic initiatives of military commands.

Following the end of war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex executive-type problems. The most common issue was - what methods should be adopted for optimum deployment of limited resources so that total profit is maximised and or total cost minimised ?

The impact of OR can be felt in many areas now a days. A large number of management consulting firms are engaged in OR activities currently. OR activities include

- Transportation system
- Hospitals
- Libraries
- Financial Institutions etc. apart from military and business applications.

While making use of the OR techniques, a mathematical model of the problem is formulated. Actually this model is a simplified representation of the problem. Then an optimal solution is found by utilising various applicable mathematical operations. Since the model is an idealized form of exact representation of real problem, most favourable solution thus obtained may not be the best. Although, at times, it may be possible to develop, extremely accurate but complex mathematical models yet they may not be able to provide the additional benefit. So from both the mathematical simplicity and cost minimising point of view, it seems good to develop a simpler model. Thus the weaknesses in the initial solution are definitely used to suggest improvements in the model. A new solution is obtained and the process is repeated until the further improvements in the succeeding solutions become so small that it does not seem economical to make further improvements.

If model is formulated carefully and tested, the resulting solution should be a good approximation of real problem. Although we may not get the best answers but definitely we are able to find some what good answers rather than not having any.

8.2 Definitions

There are several definitions of OR given by researchers

Year	Researcher Name	Definition
1946	Morse and Kimbal	OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
1948	P.M.S.Blackett	OR is a scientific method of providing executive with an analytical and objective basis for decisions.
1948	P.M.Morse	OR can be considered to be an attempt to study those operations of modern society which involved organization of men or of men and machines
1957	Churchman, Acoff, Arnoff	OR is the application of scientific methods, techniques and tools to problem involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problem.

1958	T.L.Saaty	OR is the art of giving bad answers to problems to which otherwise worse answers are given.
1968	Jagjit Singh	OR is a management activity pursued in two complementary ways- one half by the free and bold exercise of commonsense untrammelled by any routine and other half by the application of a repertoire of well established procreated methods and techniques.
1968	Ackoff and Sasieni	OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized systems so as to provide solutions which best serves the purpose of the organization as a whole.
1975	Thiearf and Klekamp	OR utilizes the planned approach and an inter-disciplinary team in order to represent complex functional relationships as mathematical models for purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis

Most of the definitions of Operations Research are not satisfactory due to three main reasons:

1. Operations Research is not a science like other scientific well defined phenomena i.e. well defined physical, biological, social phenomena. Chemists know about atoms and molecules and have theory about their interactions; biologists know about living organisms and have theory about their vital processes, operations researchers don't claim to know or have theories about operations. It is essentially a collection of mathematical techniques and tools which in conjunction with a system approach are applied to solve practical decision problems of an applied nature.
2. Operations Research is inherently inter-disciplinary in nature and has applications in military, business, medicine, engineering and so on. OR makes use of expertise and experience of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of OR which is not included in most of its definition.
3. Most of the definitions of OR have been offered at different times of development of OR and hence these emphasize only one or the other aspect.

8.3 Characteristics

Various definitions of OR present different characteristics of OR. These are following:

i. Application of scientific methods
ii. Use of inter-disciplinary teams
iii. System orientation
iv. Human factors
v. Quantitative solutions
vi. Use of computers
vii. Uncovering of new problems
viii. Improvement in the quality of decisions

8.4 Why Operations Research?

After studying in details about Operation Research, its characteristics, its need in industries canbe studied. Following factors contribute to the need and utility of Operations Research.

a) **Complexity:** In the present world, the number of factors influencing a decision has increased drastically. Situations are becoming more complex and big because these factors are influencing each other in a very complex manner. Hence, there is a great uncertainty about outcome of interaction of factors like technological, environmental, competitive etc. Take an example of an industry process i.e. production schedule which is taking into account:

   a. Customer demand in critical situation.
   b. Raw material requirement and availability.
   c. Equipment capacity and probability of equipment failure.
   d. Restriction on various manufacturing processes.
   e. Number of operators/employees absent.
   f. Power cut on that day.

   Evidently, it is not easy to prepare a schedule which is both economical and realistic by considering above factors without the help of some scientific technique. Thus, there is a need of mathematical model which, in addition to optimization helps to analyse the complex situation.

b) **Scattered responsibilities and authority:** In the modern world when sizes of industries are very big and responsibilities and authorities of decision makers are scattered throughout the organizations and thus the organization if it is not conscious, may be following inconsistent goals. Mathematical quantification or OR overcomes this difficulty to a great extent.

c) **Uncertainty:** There is a great uncertainty about economic and general environment. With economic growth, uncertainty is also increasing. This makes decision making more difficult, costlier and time consuming for the systems. OR, is, thus, quite essential from reliability point of view.

d) **Knowledge explosion:** The knowledge is increasing at various levels at a very fast rate. Majority of the industries are not up-to-date with the latest prospects and are, therefore, at a disadvantage. Various OR teams working for the industries gather latest information for analysis purposes which is quite useful for the industries.

8.5 Applications of Operations Research Techniques

OR is being widely used in various types of industries, businesses, Governments, military, agricultural & transportation sectors. A few industries / industrial sectors can be named such as: automotive, airlines, railways, electronics, media, coal, petroleum, paper, chemical, software, etc. Few of the techniques which are successfully applied are:

- **Inventory control:** This model is used to sort out problems related to stocks by finding out Economic Order Quantities, safety stocks in the inventories, minimum and maximum order quantities per lot and availability of minimum and maximum stocks.

- **PERT & CPM network techniques:** These programmes are used in various types of projects like flyovers, railway tracks, dams, highways, roads for planning and scheduling of activities, also in development and production of ships, aircrafts, helicopters, computers etc.

- **Queuing theory:** This theory is used in solving problems which are related to scheduling of air traffic, repair and maintenance of automotive industrial areas, bank counters, bus and railway bookings, traffic congestion etc.

- **Linear Programming:** This technique is widely used in solving the assignments of job allocation to machines, materials, distribution, transportation etc.
- **Simulation:** The theory being used in analyzing market situations where probability factors are linked to sort out problems.
- **Dynamic Programming:** It is applied in areas like selection of advertising media, cargo loading and distribution, routing optimization etc.
- **Decision Theory:** It has been used in analysing and finding methods for making decisions under various situations of uncertainty and risk to meet the stated objectives.

8.6 Operations Research Models

A model in OR is used as a method of representation of an actual object or a situation. A model shows relationships (directly or indirectly) and inter-relationships of action and reaction in terms of cause and effect.

The main objective of a model is to provide a means for analyzing the behaviour of the system for the purpose of improving its performance. In case, the system existence is futuristic then a model defines the structure of this future system indicating the functional relationships of various elements in the system. A model allows the examination of behaviour of a system without interfering with ongoing operations.

There are various characteristics which are the base for classification of models, these can be classified as:

- **Classification by structure:**

  i. Iconic models: It represents the system as it is by scaling up or down (i.e. by enlarging or reducing the size). In other words, it is an image. For e.g. a toy tank is an iconic model of a real one. Other examples can be photographs, drawings, maps etc. Similarly 'a globe'; the diameter of earth is scaled down but the shape of earth is approximately the same. Relative sizes of continents, seas, rivers, mountains etc. are approximately correct.

  ii. Analogue models: In this type of model, one set of properties is used to represent another set of properties. After solving the problem, the solution is reinterpreted in terms of the original system. For e.g. graphs are very simple analogues because distance is used to represent the properties such as: time, number, percent, age, weight and many other properties.

  iii. Symbolic (Mathematical) models: It employs a set of mathematical symbols (i.e. letters, numbers etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour of the system.

- **Classification by Purpose :**

  i. Descriptive models: A descriptive model describes some aspects of a situation based on observations, survey, questionnaire results or available data. For example: the result of an opinion poll.

  ii. Predictive models: In this type of model, "what if" type of questions can be answered, i.e. they can make predictions regarding certain events. For e.g. based on survey results, television networks using attempt to explain and predict the election results beforehand.

  iii. **Prescriptive models:** Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive model because it prescribes what the managers ought to do under the given circumstances.

- **Classification by nature of environment :**

  i. Deterministic models: These models assume conditions of complete certainty and perfect knowledge. For e.g. transportation & assignment models are deterministic types of models.

  ii. Probabilistic (or Stochastic) models: These models usually handle the situations in which the consequences of managerial actions cannot be predicted with certainty. For e.g. insurance companies insure against risk of fire, accidents, sickness etc.

- **Classification by Behaviour:**

  i. Static models: These models don't consider the impact of changes that takes place during the planning horizon i.e. they are independent of time. Also, in a static model only one decision is needed for the duration of a given period of time.

  ii. Dynamic models: In these models, time is considered as one of the important variables and admits the impact of changes generated by time. Also, in dynamic models, not only one but a series of independent decisions is required during the planning horizon.

- **Classification by Method of solution:**

  i. Analytical models: These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For e.g. a general Linear Programming model and specifically structured transportation models are analytical models.

  ii. Simulation models: These models also have mathematical structures but they cannot be solved by purely using the tools and techniques of mathematics.

- **Classification by Use of Digital Computers:**

  i. Analogue and mathematical models combined: Sometimes analogue models are also expressed in terms of mathematical symbols. For e.g. simulation model is of analogue type but mathematical formulae are also used in it.

  ii. Function models: These models are grouped on the basis of the function being performed.

  iii. Quantitative models: These models are used to measure the observations.

  iv. Heuristic models: These models are mainly used to explore alternative strategies, or arrive at some acceptable decision by using some rule or procedure. Such rules may not have sound mathematical proofs to justify their use, but have been found to be useful in practice.

8.7 Principles of Modelling

Some principles are discussed which are useful in providing guidelines during when formulation of models takes place. While model building, these principles can be kept in mind:

1. **Better to make a simple model than a complicated one:**

   It is a common guiding principle for mathematicians who are in process of building a strongest possible model that has wide applications.

2. **Tendency to mould the problem to fit the technique:**

   For example – an expert on linear programming techniques may tend to view every problem he encounters requiring a linear programming solutions. In fact not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. Therefore, not all real-world problems call for Operations Research. Of course everyone searches and depicts reality in his own terms, so the field of OR is not unique in this regard.

3. **Beware of over-selling a model:**

   This principle is of importance for the OR professional as most of the non-technical benefactors of an operations researcher's work are not likely to understand his methods. More technicality in the method also increases the responsibilities and burden on the OR professionals to distinguish clearly his role; as model manipulator and model interpreter. In those cases, where the assumptions can be challenged, it would be dishonest to use the model.

4. **A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended:**

   One example of this error can be taken as the 'use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.

5. **A model should never be taken too literally:**

   For example, there is a requirement of sophisticated model for Indian economy and a lot of researcher teams are working on the project and these teams are spending a great deal of time and expenses to gather all information regarding the complicated relationships amongst factors. Under these circumstances, it can be believed that the model is an excellent representation of the real system, thus may have been in use under the misplaced impression that so much of time effort must have resulted in a' good model'.

6. **Models should be validated prior to implementation:**

   Taking example of monthly sales forecast of a particular commodity, it can be compared with historic data for testing purpose. If there is no possibility of validation testing before it is implemented then it can be implemented in phases for validation. We can understand by taking a new model for inventory control, first the model can be implemented with few selected items while the older system is retained for the majority of remaining items. If the model proves to be successful, more items can be taken.

7. **The deduction phase of modelling must be conducted rigorously:**

   When conclusions of the model come into the picture and they are inconsistent with reality, it means there is some error in the assumptions which are being taken hence rigorous deduction of modelling is necessary. One application of this principle can be discussed i.e. Programming of computers must be done very carefully, Hidden "bugs are very dangerous as they don't prevent the programmes from running but simply produce results which are not consistent with the intention of the models.

8. **Some of the primary benefits of modelling are associated with the process of developing the model:**

   It is important to know that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself is non-explanatory and does not provide itself full knowledge of the system that the modeller should acquire in order to successfully build it. In some cases, the sole benefit of the modelling occurs when it is being built and after completion of the model, it has no further value.

9. **A model cannot be any better than the information that goes into it:**

   A model can only manipulate data as in case of a computer also, which is provided to it. Models can condense data or they can use the data to provide some more useful information or data but they don't have the quality or capacity to generate it.

10. **Models cannot replace decision makers:**

    It should not be understood that model can be used to provide 'optimal solutions free from human subjectively. OR models can aid decision makers and therefore provide a lot of support to take better decisions to be made. Undoubtedly, experience, intuition and judgement in decision making are undiminished.

8.7.1 Characteristics of a Good Model

A good model should possess the following characteristics.

- It should be easy and economical to construct.
- The number of simplifying assumptions should be as low as possible.
- It should be adaptable to parametric type of treatment.
- The number of relevant variables should be as low as possible. This means the model should be simple yet close to reality.
- It should assimilate the system environmental changes without change in its framework.

8.7.2 Advantages of Model in OR

Following are the advantages of using model in Operation Research.

- It helps in finding avenue for new research and improvements in a system.
- It indicates the scope as well as limitations of a problem.
- It provides a logical and systematic approach to the problem.
- It represents the overall structure of the problem more comprehensible and helps in dealing the problem entirely.

8.7.3 Limitation of Model

Following are the limitations of a model.

- The validity of a model for a particular situation can be ascertained only by conducting experiments on it
- Models are only a representation of reality and should not be regarded as absolute in any case.

8.8 Constructing a Model

Construction of a model is multi-step process. In first step, formulation of the problem requires analysis of the system under study. This analysis shows various phases of the system and the way it can be controlled. With the formulation of the problem, the first stage in model construction is over. In next step, 'measure of effectiveness' is defined i.e. the next step is to construct a model in which effectiveness of the system is expressed as a function of the variables defining the system. The general form of an OR model is:

$$E = f(x, y),$$

Where E = Effectiveness of the system

x = Variable of the system that can be controlled

y = Variable of the system that cannot be controlled

Deriving of solution from such a model consists of determining those values of control variable x, for which the measure of effectiveness is optimized. Optimization includes both maximization (in case of profits, production capacity etc.) and minimization (in case of losses, cost of production etc.).

Various steps in the construction of a model are:

- **Selection of components of the system:** All the components of the system which contribute towards the effectiveness measure of the system should be listed.
- **Pertinence of components:** Once a complete list of components is prepared, the next step is to find whether or not to take each of these components into account. This is determined by finding the effect of various alternatives courses of action on each of these components.
- **Combining the components:** It may be convenient to group certain components of the system. For e.g. material price, transportation charges of material and inspection cost of raw material can be grouped together and called "raw material acquisition cost". The next step is to find out, for the components in modified list whether its value is fixed or variable.
- **Substituting symbols:** Once each variable component in the modified list has been broken down like this, symbols may be assigned to each of these sub-components.

8.9 Types of Mathematical Models

There are various types of mathematical models which are given below

- **Statistical Techniques:**

  Some of the techniques used in various functions, areas in this world come from Statistics & Probability theories. A lot of factors in various areas cannot be predicted in absolute certainty hence probability concept is associated with those factors to analyse the uncertainties in the situation and data is used with some reasonable accuracies to reach some decision.

- **Queuing Models:**

  These models (may be used where the situations) involve the arrival of units to be serviced at one or more stations. We can take example of car or any other vehicle reaching at loading station, patients arriving any dental or other clinics, aeroplane reaching aerodromes, parcels reaching at P.A. desk, customers entering a fair or a departmental store and so on. In case of single service station, there will be a single queue and multiple queues can be used in multiple service stations (e.g. in a barber shop). The same principle is applied in case a job is done through various processes on different machines in workshop. In queuing model, there may be more demand of the material or less at subsequent operation. The cost of waiting and idling of the material is different and needs to be calculated and optimum queuing model to be selected to minimize the cost and improve the process.

- **Simulation Techniques:**

  This technique is a data generating technique which actually does not solve a problem but it generates data or the information which is helpful in taking decisions. This method or technique is used in cases when it is very much time consuming process to conduct an experiment or do actual real study to find exact situation of a problem. Also, sometimes when there are large no. of variables are available and there is lot of complexity between various variables. There is no possibility to develop an analytical model representing the situation. For an example, there are two ways of developing the project, the designer can make various aerodynamics equations of the plane and solve it and the second way is to make a model of the aeroplane can be prepared and tested in a windmill.

Simulation is a very powerful method and one of the most widely used technique of operations research. There are so many important decision-making problems which are too intricate to be solved by mathematical analysis and the experimentation with the actual systems.

- **Competitive Models:**

  When two or more organizations or individuals are making decisions with conflicting objectives. In these circumstances when one makes some decision, it will affect the decision made by another individual(s). These models are applicable to situations like two players trying to win a game, planning of war situations by two opponents, fight for a political election for a seat between two candidates, companies trying to improvise share hold in the market etc.

- **Allocation Models:**

  These are used to solve the problems when there are so many jobs to be performed and there are lot alternatives to complete the job at different stages with availability of resources are limited. Simplest allocation (assignment) model is that in which no. of allocation of jobs is same as that of no. of resources (men). The assignment problem becomes more complex when some of the jobs require more than one resource or if the resource can be used for more than one job.

- **Decision Theory:**

  Decisions are part of routine for mangers in organisations. The wrong decisions can create blunders for the organisations and similarly right decisions may change / increase the profits marginally. There may be two categories of decisions Strategic and Tactical decisions. Strategic decisions affect the organisations in the long run and Tactical decisions affect the organisation in short term. New product range development, site allocation, large scale automation etc. are the long term strategic decisions.

  Before the decisions are taken by the manager, the data regarding the subject is collected, formulation of data is done and all the possible alternatives are taken into account. Decision theory can make use of probability factors of various outcomes. For example, a television manufacturing unit wants to expand the production capacity, it can take various alternatives into consideration and select one out of those. There may be following alternatives:

    i. Increase capacity of the existing plant,
    ii. Build a new production site,
    iii. Hire production shop of other television company.

  The main factor or uncertainty in the above situation is future demand of the television which can be out of the following:

    i. High demand
    ii. Moderate demand
    iii. Low demand
    iv. No demand

  Decision theory covers three categories of decision making:

i. Decision making under the conditions of certainty
ii. Decision making under conditions of risk
iii. Decision making under the conditions of uncertainty

- **Replacement Models:**

There are lot of equipments, machinery, and parts being used by industries, railway and other organisations like military etc. These machines and equipments get deteriorated with time and the efficiency decrease with span of time. Running machines require more operational cost, running vehicles need more maintenance costs. When the cost of maintenance and repair becomes more expensive, it becomes necessity to replace the parts or the machines completely. There are no fixed times when the replacements are needed**.** The replacement models help in making replacement policies to calculate increased costs of repair, maintenance, scrap value and cost of equipment to find the right intervals of replacements.

Similarly, there are few items like bulbs, tube-light etc. Which do not deteriorate but fail without giving an alarm? The problem here is to find out which item to be replaced and at what intervals, they should be replaced individually or in groups, the cost of handling, cost of replacement and cost associated with failure of items to be calculated.

There is one more condition when the part needs replacement due to new inventions. In this situation, the equipment is being replaced not because of 'not longer' performance to the design standards but due to because modern equipment performs to higher standards.

- **Sequencing Models:**

These models are used in situations in which the effectiveness measure such as time, cost or distance is a function of the order or sequence of performing a series of tasks. The selection of the appropriate order in which waiting customers may be served is called sequencing. For e.g. x no. of machines are used to do Xn type of operations and there are y no. of jobs to be allocated on machines. Now the problem is to find out right sequence of operations so as to minimize total elapsed time on the machines with condition that machines should be idle for minimum time.

- **Activities Scheduling by PERT & CPM:**

There are cases when the no. of interrelated are huge in complex and big projects. In these projects, management does not give the schedules of completion and in process just by intuition. Therefore, some techniques are used by management teams for planning, scheduling and controlling the projects. CPM and PERT are two techniques which are used to show diagrammatically representation of various activities, their relations, dependencies etc. Time completions of these activities are shown by arrows. There may be exact completion times of activities or probability of completion of those activities. Depending on need and availability of data, different techniques are used.

- **Routing Models:**

Routing problems are directly or indirectly related with sequencing. There are two important routing problems:

✓ The travelling salesman problem

✓ The minimal path problems

In the travelling salesman problem, there may be a no. of cities where salesman will visit. The distance between each cities is known, problem is to find out shortest distance of travel when the person will start travelling from his home, go to different cities and come back to home city after travelling different cities.

- **Inventory Models:**

  These models take data and experience of idle resources such as raw material, stock material, machines, men, money etc. Models are concerned with two decisions:

  - ✓ How much quantity must be ordered &
  - ✓ When the order should be placed

  So that the production should not stop, material shortages should not be there. Cost of ordering and transportation should be minimized. Inventory should not go beyond minimum and maximum levels. Cost of releasing order, follow up costs, storage space cost, capital and interest on capital cost should be tried to minimize.

- **Reliability Theory:**

  Reliability is a factor which defines quality performance of the equipment, machine or a part. Reliability is the capability of an equipment to perform without failure or malfunctioning for a specific time period under given circumstances or environment. The equipment can be any simple one like fan, electric heater, pulley, motor or any complex one like computer or an aeroplane.

  Reliability of a machine or equipment depends upon the reliability of various parts of which it is made. It is necessary that the equipment must perform the functions for which it is made. Otherwise, the cost of failure can be very high and hazardous depending upon the use where it is being used. For e.g. malfunctioning of software in space or in aeroplane during flight is unthinkable.

  The reliability factor in quantitative term can be obtained for individual component by testing procedures i.e. life testing. The reliability of the equipment can then be ascertained by using the reliability of individual components and the design of equipment which indicates the interconnection of components.

- **Advanced OR models:**

  - ✓ **Non-linear Programming:**

    If objective function and / or one or more of the constraints are non-linear in nature then non-linear programming is used. Factors such as price discounts on bulk purchases, graduated income tax etc. may cause non-linearity in the model. The non-linearity of the functions makes the solution of the problem much more involved as compared to linear programming problems and there is no single algorithm like simplex method which can be employed for situation of the problem. Though, a number of algorithms have been developed, each is applicable to specific type of non-linear programming problem only.

  - ✓ **Integer Programming:**

    The method is used to solve the problems in which some or all variables must have non-negative integer values. When all the variables are constrained to be integers, the problem is called integer programming problem and in case only some of the variables are restricted to have integer values, the problem is said to be a mixed integer programming problem. In some situations, each variable can take on the values of either zero or one as in 'do' or 'not do' type decisions. Such problems are referred to as zero-one programming problems.

Examples of integer programming are: number of trucks in a fleet, number of generators in a power house. Approximate solutions can be obtained without using integer programming methods but the approximation generally becomes poor as the values of the variables become smaller.

✓ **Dynamic Programming:**

These models are used to make a series of interrelated decisions for multi-stage problems that extend over a number of time periods. For example, a company may wish to make a series of marketing decisions over time which will result in higher possible sales. Principle of the model is that regardless of what the previous decisions are it tries to determine the optimum decisions for the periods that still lie ahead. The dynamic programming approach divides the problem into a no. of sub problems or stages. Though Dynamic programming models are used for the problems in which time plays an important role, yet in many dynamic problems, time is not a relevant variable. For instance, suppose a company has decide. 'capital C' to be spent on its advertisement of product and three advertising media can be used for the advertisement i.e. radio, television and news paper. In each media, advertisement can appear a number of times per week. Each appearance in the advertisement constituents some cost and the returns w.r.t advertisement. Hence the problem is to determine, how many times, an advertisement should be appeared in different alternate media per week so that the returns are at the maximum and costs are within prescribed limits.

✓ **Goal Programming:**

These models are similar to linear programming but are used in situations where there are multiple goals and objectives. A company which is manufacturing lathes and milling machines, there may be following objectives:

i. Maximize profits,
ii. Maximize number of lathes to be manufactured,
iii. Maximize number of milling machines to be manufactured.

It is obvious that all three goals cannot be added since units of measurement are decides targets for each of the objective and ranks them in order of their importance. Receiving this information, goal programming tries to minimize the deviations from the targets. It starts with the most important goal and continues till the achievement of a less important goal.

✓ **Heuristic Programming:**

This programming uses the rules of thumb to explore the most likely paths and to make educated guesses to arrive at solution of a problem. This models seems to be quite promising for future OR work. They bridge gap between strictly analytical formulations and the operating principles which the managers are habitual of using. They go with step-by-step approach towards the optimal solutions when a problem cannot be solved by mathematical programming form.

✓ **Quadratic Programming:**

This refers to the problems in which objective function is quadratic in form (contains square terms) while the constraints are linear. A number of efficient algorithms have been developed for such problems.

✓ **Sensitivity Analysis:**

After attaining optimal solution to a linear programming model, it may be desirable to study how the current solution changes when the parameters of the problem get changed.

✓ **Parametric Programming:**

The study of the effect of continuous changes in the values of the parameters on the optimal solution to a linear programming problem.

- **Combined Methods:**

  Real systems may not involve only one of the models which are explained in above paragraphs. For example, in production related problems, normally a combination of inventory, allocation and queuing models are applied.

  These combined models are solved once at a time and in some logical sequence. However, In Operations Research, these models are combined and with the combination of those models, Master model is built which considers many individual models.

  At the last, it can be said that above mentioned / explained models solve a lot of problems in OR but all the problems cannot be covered by these. It is also possible in near future, more new processes will be revealed and subjected to more mathematical analysis.

- **Mathematical Techniques:**

  There is possibility that any mathematical technique can become a useful tool for operations analysis. Most commonly used mathematical techniques may be used such as integral equations, differential equations, vector and matrix theory etc.

8.10 Management Applications of Operations Research

Some of the areas of management of decision making where the tools and techniques of OR are applied can be outlined as follows:

i. Accounts – Investments and Budgeting :

a. Cash flow analysis, long range capital requirements, dividend policies, investment portfolios
b. Credit policies and risks, delinquent account procedures
c. Claim and complaint procedures

ii. Purchasing, procurement and exploration :

a. Rules of buying, supplies; stable or varying prices
b. Knowing quantities, quality and times of
c. Replacement policies
d. Defining strategies for exploration of raw material sources

iii. Production Management :

a. Physical distribution
   i. Location and sizes of warehouses, distribution centres and retail outlets
   ii. Defining distribution policy
b. Facilities planning
   i. Numbers and location of factories, warehouses, hospitals etc.

   ii. Loading, unloading facilities for trucks, rails and determining transportation schedules

c. Manufacturing
   i. Production scheduling and sequencing etc.
   ii. Stabilizing production, employment training and layoffs

d. Maintenance and project scheduling
   i. Maintenance policies and preventive maintenance.
   ii. Project scheduling and allocation of resources

iv. Marketing:

a. Product selection, competency actions
b. Number of salesman, frequency of calling on account percent of time spent on prospects
c. Advertising media w.r.t. cost and time

v. Personel Management :

a. Selection of suitable personnel on minimum salaries
b. Mixes of age and skills
c. Recruitment policies and assignment of jobs

vi. Research and Development :

a. Determination of the areas of concentration of research and development
b. Project selection
c. Determination of time cost trade off and control of development projects
d. Reliability and alternative design

8.11 Scope of Operations Research

In the recent years of organized development, OR has entered successfully in many areas of research for industry, government, education and military. Basic problem in developing countries is to remove poverty as soon as possible. So there is great scope for administrator, economists, politicians and technicians working in a team to solve problem by using OR technique. OR is also useful in the following fields:

1. In Defence:

OR has a lot of scope in defence operations. In modern phase, defence operations are divided into Army, Airforce and Navy and further it is subdivided into operations, intelligence, administration, training and the others. Therefore, to coordinate various departments and to arrive at optimum strategy, OR is useful.

2. In Production Management:

Production manager can use OR techniques to:

- Schedule the type of machine to be used for producing various items
- Decide no. and size of items to be produced
- Calculate optimum product mix
- Select, locate and design the sites for production plant

3. In Marketing:

Marketing manager can decide:

- Where to distribute product so that cost of transportation is optimized to minimum
- Selection of right advertising media wrt time

4. **In Industry:**

   There are a lot of processes through which material is passed in industries, starting from purchasing of the material to various manufacturing processes and then despatch. The management is interested in optimizing the costs of various processes to maximize profits. OR team studies all these processes, compares the alternatives and fixes the right ones. OR has been used in fixing in various fields of production, sales, purchases, inventory control, transportation, scheduling, sequencing, planning, maintenance and repairs.

5. **In Agriculture:**

   OR approach needed to be equally developed in agricultural sector nationally and internationally. With increasing population all over and shortage of food due to unplanned behaviour, OR techniques application was required. The problem of optimal distribution of water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

6. **In Personnel Management:**

   Personnel manager can decide various activities using OR techniques:

   - Optimizing suitable no. of persons in minimum salary
   - Determining best age for retirement for the employees
   - To decide no. of persons to be appointed on full time basis for seasonal workload

7. **In Finance:**

   In the modern world, it is necessary for every Government to utilise its resources for economic development of the country:

   - To select best profitable plan of the company or the institution
   - To maximize per capita income

8.11.1 Role of Operations Research in Decision Making

Operations Research may be regarded as a tool which is used by the management to increase the efficiency in the operations and effectiveness of their decisions. OR techniques can be used to find out location of the factories, projects, and location of warehouses, transportation means and their tracks. The advantages of usege of OR techniques in business and systems may be classified as:

1. **Better control:** It is very costly for big concerns management to provide routine decisions through a lot of supervisors. OR approach helps executives to devote attention to more pressing situations. OR approach helps in production scheduling and inventory management.

2. **Better co-ordination:** Sometimes OR has been useful in maintaining law and order situation out of chaos. OR based planning becomes driving force in alarming situation of marketing decisions with the limitation of manufacturing capabilities.

3. **Better system:** OR techniques are useful in analyzing and deciding warehouse establishment out of various alternates.

4. **Better decisions:** OR models frequently yield actions that do help in intuitive decision making. Sometimes human mind cannot incorporate all the important factor without the help of OR and computer analysis.

8.12 Development of Operations Research in India

Existence of Operations Research in India is considered in year 1949 when a branch of OR was setup in Hyderabad. At the same time, Prof. R.S.Verma from Delhi University setup an OR team in Defence Science Laboratory. Prof. P.C.Mahalanobis established an OR team in Indian Statistical Institute, Kolkata to sort out National survey and planning related problems. In year 1957, Operations Research society of India was formed and the same society became a member of International Federations of Operations Research Societies in year 1960. A lot of Journals are published by / in India like 'OPRESEARCH', 'Industrial Engineering and Management', 'Material Management Journal of India', 'Defence Science Journal', 'SCIMA'.

University of Delhi was the first one to start OR education in India, which started M.Sc. in Operations Research in 1963. Simultaneously, OR was added as a subject in MBA in Institute of Management at Kolkata and Ahmadabad. In the recent years OR has become so much popular that it has been introduced in almost all Institutes and Universities in various disciplines like Mathematics, Statistics, Commerce, Economics, Management studies, Medical Science etc.

8.13 Role of Computers in Operations Research

Computers have played a very important role in the development and progress of Operations Research in India. There are so much complex computations in Operations Research that without the help of computers it can take weeks and months to yield the results manually. So the computers have become very essential and integral part of OR. These days, OR and Computers methodology are developing simultaneously.

Computer software packages are being used for effective and rapid calculations which is necessary part of OR approach to solve various complex problems. Some of the softwares are:

- **Quantitative System for Operations Management:** This software is very user friendly and it, itself has problem solving algorithms for operations management problems.
- **Quantitative System for Business Plus, version 3.0:** This software can be used for solving algorithms, doing basic statistics, non-linear programming and financial analysis for Operations Research.
- **LINDO (Linear Interactive Discrete Optimization),** developed by Linus Scharge Lindo.
- **Value STORM: MS Quantitative Modelling for Decision support:** This is a special version of Personal STORM version 3.0 developed to be used for Operations Research.

8.14 Summary

In this unit, you have studied what Operations Research is today. It also discusses what are the basic reasons that most of the definitions of OR are not satisfactory .What are the characteristics of it and why it was needed in industry? It also describes the various application area of Operations Research.It also discusses about OR Model, taxonomy of OR model, its pros and cons and procedure of constructing an OR Model. What are the characteristics a good model should possess.It illustrates the management application of OR, its scope and its role in decision making. What are the recent developments of it in India and also role of computers in operations research.

8.15 Key Words

- **Operations Research (OR):** It is a scientific method of providing executive with an analytical and objective basis for decisions.
- **Model** A model is used as a method of representation of an actual object or a situation. A model shows relationships (directly or indirectly) and inter-relationships of action and reaction in terms of cause and effect.
- **Programme Evaluation & Review Technique (PERT) & Critical Path Method (CPM):** are two techniques of OR which are used to show diagrammatic representation of various activities, their relations, dependencies etc.
- **Linear Programming (LP):** LP technique in OR is widely used in solving the assignments of job allocation to machines, materials, distribution, transportation etc.
- **LINDO (Linear Interactive Discrete Optimization): Computer software:** Computer software package which is used for effective and rapid calculations.

8.16 Self Assessment Test

1. What is Operations Research? Why it was needed in Industry?
2. What are the characteristics of OR? How it plays an important role in an organisation for decision making?
3. Explain various applications of OR.
4. Explain different types of models in OR, their characteristics, advantages and limitations.
5. Is there any use of computers in OR techniques. Elaborate.
6. Write a short note on

   a) Scope of OR

   b) Developments of OR in India
7. Describe the various reasons that OR definitions are not satisfactory.
8. What do you mean by OR Model? What are the principles of modelling? Describe the process of constructing an OR Model.

8.17 References

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Unit - 9 : Linear Programming and Transportation Models

Unit Structure:

9.0 Objective

After studying this unit, you should be able to understand

- General Formulation of Linear Programming problem
- Standard & Matrix form of LP problems
- Application of Linear Programming
- Limitations & advantages of Linear Programming Technique
- Transportation Problems & its Matrix form
- Tabular Representation of a Transportation Model
- Solution of transportation problem

9.1 Introduction

In mathematical expression, a linear form can be defined as $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$ where $a_1, a_2, \ldots, a_n$ are constants and $x_1, x_2, \ldots, x_n$ are the variables. The theory was observed by George Dantzig and his associates in 1947 while working in U.S. Air Force department that a large no. of military programming and planning problems could be formulated as maximising / minimising a linear form of profit / loss function whose variables were restricted to values satisfying a system of linear constraints. The term programming may be defined as a process of determining a particular programme or plan of action.

9.2 Conditions

Linear programming can be considered as one of the most important techniques developed in the field of Operations Research (OR). Linear Programming can be used for optimization problems if the following conditions are satisfied

1. A well defined objective function must be there (such as profit, cost or quantities produced) which is to be either maximised or minimised and which can be considered as linear function of decision variables.
2. There must be constraints to the extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities in terms of variables.
3. Alternate courses of action must be there. To understand through an example, an automotive part can be produced on two different drilling machines, now problem is how much quantity must be produced on different machines.
4. Decision variables should be interrelated to each other and must be non-negative. The non-negativity condition of the variables shows that they are real life problems seeking solution.
5. The resources must be in limited supply.

9.3 General Formulation

The general formulation of the linear programming problem can be stated as

In order to find the values of n decision variables $x_1, x_2, \ldots, x_n$ to maximize or minimize the objective function

$$Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots + c_n x_n$$

And also satisfy m constraints

$$\left.\begin{array}{l}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1j} x_j \ldots\ldots\ldots\ldots + a_{1n} x_n \,(\leq \text{ or } \geq)\, b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2j} x_j \ldots\ldots\ldots\ldots + a_{2n} x_n \,(\leq \text{ or } \geq)\, b_2 \\
\\
a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{ij} x_j \ldots\ldots\ldots\ldots + a_{in} x_n \,(\leq \text{ or } \geq)\, b_i \\
\\
A_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mj} x_j \ldots\ldots\ldots\ldots + a_{mn} x_n \,(\leq \text{ or } \geq)\, b_m
\end{array}\right\}$$

Where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions

$$x_1 \geq 0, x_2 \geq 0, \ldots\ldots\ldots, x_j \geq 0, \ldots\ldots\ldots\ldots, x_n \geq 0.$$

However, by convention, the values of right side parameters b_i ($i = 1, 2, 3 \ldots\ldots\ldots\ldots, m$) are restricted to non – negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constant by -1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

9.4 Slack & Surplus Variables

Slack variables If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to the left hand side.

The non negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

$x_1 + x_2 \leq 2,$ ....(1)

$2x_1 + 4x_2 \leq 5,$ ....(2)

$x_1, x_2 \geq 0$

We add the slack variables $x_3 \geq 0$ & $x_4 \geq 0$ on the left hand sides of above inequalities resp., Which gives

$x_1 + x_2 + x_3 = 2,$ ....(1)

$2x_1 + 4x_2 + x_4 = 5,$ .... (2)

$x_1, x_2, x_3, x_4 \geq 0$

Surplus variables If a constraint has $\geq$ sign, then in order to make it equality, we have to subtract something non-negative from its left hand side.

Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For e.g., considering the following constraints

$x_1 + x_2 \geq 2,$ ....(3)

$2x_1 + 4x_2 \geq 5,$ .... (4)

$x_1, x_2 \geq 0$

We subtract the surplus variables $x_3 \geq 0$ & $x_4 \geq 0$ on the left hand sides of above inequalities resp., Which gives

$x_1 + x_2 - x_3 = 2,$ ....(1)

$2x_1 + 4x_2 - x_4 = 5$ ....(2)

$x_1, x_2, x_3, x_4 \geq 0$

9.5 Standard Form

The standard form of linear programming problem is used to develop the procedure for solving general linear programming problem. The characteristics of the standard form are explained in the following steps

Step I All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities (≥ 0). Constraints of the inequality type can be changed to equations by adding or subtracting the left side of each such constraint by non-negative variables. These new variables are called slack variables and are added if the constraints are ($\leq$) or subtracted if the constraints are ($\geq$). Since in the case of $\geq$ constraint, the subtracted variable represents the surplus of the left side over the right side, it is common to refer to it as surplus variable. For convenience, however, the name 'slack' variable will also be used to represent this type of variable. In this respect, a surplus is regarded as a negative slack.

For example, consider the constraints

$3x_1 - 4x_2 \geq 7$, $x_1 + 2x_2 \leq 3$

These constraints can be changed to equations by introducing slack variables x_3 & x_4 resp. Thus we get

$3x_1 - 4x_2 - x_3 = 7$, $x_1 + 2x_2 + x_4 = 3$ & $x_3, x_4 \geq 0$

Step II The right side element of each constraint should be made non-negative (if not). The right side can always be made positive on multiplying both sides of the resulting equation by (-1) whenever it is necessary. For example, consider the constraint as

$4x_1 - 4x_2 \geq -4$ which can be written in the form of equation $3x_1 - 4x_2 - x_3 = -4$ by introducing the surplus variable $x_3 \geq 0$

Again multiplying both sides by (-1), we get $-3x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step III All variables must have non-integer values

A variable which is unrestricted in sign (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus if x is unconstrained in sign, it can be replaced by (x', -x") where x' and x" are both non negative, that is x', x" ≥ 0

The objective function should be of maximization form.

The minimization of a function f(x) is equivalent to the maximization of the negative expression of this function, f(x), that is,

$$\text{Min. } f(x) = \text{Max. } [-f(x)]$$

For example, the linear objective function

Min. $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Is equivalent to Max (-z) i.e. Max. $z' = -[c_1 x_1 + c_2 x_2 + \ldots + c_n x_n]$ with $z = -z'$

Consequently, in any L.P. problem, the objective function can be put in the maximization form.

By applying above mentioned steps systematically to general form of L.P. problem with all '$\leq$' constraints, the following standard form is obtained. Of course, no difficulty will arrive to convert the general LPP with mixed constraints ($\leq = \geq$)

$$\text{Max. } z = c_1 x_1 + c_2 x_{2} + \ldots + c_n x_n + 0\, x_{n+1} \ldots + 0\, x_{n+m}$$

And also satisfy m constraints

$$\left.\begin{array}{l}
\text{Max. } z = c_1 x_1 + c_2 x_2 + \dots\dots\dots + c_n x_n + 0\, x_{n+1} \dots + 0\, x_{n+m} \\
\text{And also satisfy m constraints} \\
a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + x_{n+1} = b_1 \\
a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n + x_{n+2} = b_2 \\
a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \dots\dots + x_{n+m} = b_m \\
\text{where } x_1 \geq 0, x_2 \geq 0, \dots x_n \geq 0, x_{n+1} \geq 0 \dots x_{n+m} \geq 0
\end{array}\right\}$$

i. This should be noted that the coefficient of slack variables $x_{n+1} = 0$, $x_{n+2} = 0 \dots x_{n+m} = 0$ (are assumed zero) in objective function so that the conversion of constraints to a system of simultaneous linear equations does not change the function to be optimized.

ii. Since in the case of (≥ 0) constraints, the subtracted variable represents the surplus variable. However, the name slack variable may also represent this type. In this respect, a surplus is regarded as a negative *slack.*

9.6 Matrix Form

The linear programming problem in standard form [(3.11), (3.12), (3.13)] can be expressed in matrix form as follows

Maximize $z = CX^T$ (objective function)

subject to, $AX = b$, $b \geq 0$ (constraint equation)

$X \geq 0$ (non-negative restriction)

$X = (x_1, x_2, \dots\dots x_n, x_{n+1} \dots\dots, x_{n+m})$,

$C = (c_1, c_2, \dots c_n, 0, 0 \dots\dots, 0)$ and

$b = (b_1, b_2, \dots b_m)$

$$A = \left\{\begin{array}{cccccccc}
a_{11} & a_{12} & \dots\dots & a_{1n} & 1 & 0 & \dots\dots & 0 \\
a_{21} & a_{22} & \dots\dots & a_{2n} & 0 & 1 & \dots\dots & 0 \\
 & & & & & \dots\dots & & \\
a_m & a_{m2} & \dots\dots & a_{mn} & 0 & 0 & \dots\dots & 1
\end{array}\right\}$$

Similarly, treatment can be adopted in case of mixed constraints ($\leq = \geq$), following points will make this clear.

The vector x is assume to include all decision variables, (i.e. original, slack and surplus). For convenience, x is used to represent all types of variables. The vector C gives the corresponding coefficients in the objective function. For example, if the variable is slack, its corresponding coefficient will be zero.

The Outlines of formulation of LP problems are explained with the help of these examples

Example 1. A company is making a chart to decide the minimum amount of constituents like proteins, vitamins, carbohydrates, fats etc. which a man needs on daily basis to fulfil his requirement for medical awareness. The choice is to be made from different type of foods (4 types). The yields per unit for different types of foods are explained below in the chart. Formulation of the linear programming model is required for this problem.

Type of food	Yield per unit			Cost per unit (Rs.)
	Proteins	Fats	Vitamins	
A	6	6	12	135
B	8	6	8	120
C	16	21	14	170
D	12	15	8	130
Minimum reqd.	1600	400	1400	

Sol. Formulation of L.P. Model

Let $x_{1,}\ x_{2,}\ x_{3,}\ x_4$ be the no. of units of food of type 1, 2, 3 & 4 resp.

Objective is to minimize the cost i.e.

Minimize Z = Rs. $(135x_{1+}\ 120x_{2+}\ 170x_{3+}\ 130x_4)$

Constraints are on the fulfilment of daily requirements of the various constituents

i.e. Proteins requirement $6x_1 + 8x2 + 16x3 + 12x_4 \geq 1600$

Fats $6x_1 + 6x_2 + 21x_3 + 15x_4 \geq 400$

Vitamins $12x_1 + 8x_2 + 14x_3 + 8x_4 \geq 1400$

Where each $x_{1,}\ x_{2,}\ x_{3,}\ x_4 \geq 0$

Example 2. Two types of products are manufactured by an organisation P_1 & P_2. They are sold at a profit of Rs. 4 & 6 resp. Products are manufactured on machines M_1 & M_2 Product P_1 requires to be processed for 2 min. on machine M_1 & 4 min. on machine M_2. Product P_2 is processed for 2 min. on machine M_1 & 2 min. on machine $M_{2.}$

Machines M_1 can be used for 800 min. a day & machine M_2 is available for 600 min. a day. Formulation of the linear programming model is required for this problem.

Sol. Let x_1 be the no. of products of type P_1 and x_2 the no. of products of type P_2. After studying the problem, the given information can be systematically represented in the table form as below

MACHINE	Type of products (minutes)		
	Type P_1	Type P_2	Available time (min.)
M_1	2	2	800
M_2	4	2	600
Profit per unit (Rs.)	Rs. 2	Rs. 3	

Since profit on selling P_1 is Rs.2 per unit, $2x_1$ will be profit on selling products of type A

Similarly profit on selling P_2 is Rs.3 per unit, $3x_2$ will be profit on selling products of type B

Total profit is given by $P = 2x_1 + 3x_2$

Since P_1 is manufactured on M_1 for 2 min. and P_2 on the same machine for 2 min., total no. of minutes on machine M_1 is given by $2x_1 + 2x_2$

Similarly P_1 is manufactured on M_2 for 4 min. and P_2 on machine for 2 min., total no. of minutes on machine M_2 is given by $4x_1 + 2x_2$

But machine M_1 is not available for more than 800 minutes, therefore

$$2x_1 + 2x_2 \leq 800$$

But machine M_2 is not available for more than 600 minutes, therefore

$$4x_1 + 2x_2 \leq 600$$

Also, it is not possible to produce negative results for quantities, hence

$$x_1 \text{ e” } 0 \ \& \ x_2 \geq 0$$

Hence allocation of the problem can be represented in the form

Find x_1 & x_2 such that the profit $P = 2x_1 + 3x_2$ is maximum,
subject to the conditions
$2x_1 + 2x_2 \leq 800$, $4x_1 + 2x_2 \leq 600$, $x_1 \geq 0$ & $x_2 \geq 0$

Example 3. A toy company is supplying two types of aeroplanes a basic model of plane P_1 & second deluxe model of plane P_2. Each plane of deluxe model takes twice as long as to produce as one of basic type. The company would have time to make a maximum of 2000 per day. Supply of raw material is sufficient to produce 1500 planes per day (both P_1 & P_2 combined).The deluxe model requires a high quality wheel set of which there are 600 sets available per day only. They are sold at a profit of Rs. 3 & 6 per piece resp. Then how many of planes of each type should be manufactured to maximize the total profit.

Formulation of the linear programming model is required for this problem.

Sol. Let x_1 be the no. of planes of type P_1 and x_2 the no. of planes of type P_2. Let plane P_1 requires t hrs, then P_2 requires 2t hrs. After studying the problem, the given information can be systematically represented in the table form as below

Since profit on selling P_1 is Rs. 3 per unit, $3x_1$ will be profit on selling planes of type P_1

Similarly profit on selling P_2 is Rs. 5 per unit, $5x_2$ will be profit on selling products of type P_2

Total profit is given by $P = 3x_1 + 5x_2$

Since P_1 is manufactured for t hrs. and P_2 2t hrs. min., total time for manufacturing is given by $tx_1 + 2tx_2$

But maximum time production is 2000 hrs, therefore

$$tx_1 + 2tx_2 \leq 2000t$$

$$x_1 + 2x_2 \leq 2000$$

But wheels sets are not available for more than 600 for deluxe model, therefore

$$x_2 \leq 600$$

Also, it is not possible to produce more than 1500 quantities, hence

$$x_1 + x_2 \leq 1500$$

Also, it is not possible to produce negative results for quantities, hence

$$x_1 \geq 0 \ \& \ x_2 \geq 0$$

HENCE allocation of the problem can be represented in the form

Find x_1 & x_2 such that the profit $P = 3x_1 + 5x_2$ is maximum,
subject to the conditions
$x_1 + 2x_2 \leq 2000$
$x_1 + x_2 \leq 1500$
$x_2 \leq 600$
$x_1 \geq 0 \ \& \ x_2 \geq 0$

Example 4. A farmer has 100 acre of farm for harvesting. potatoes, carrots & beans produced by him are sold by him at the rates Rs. 2.00, Rs. 1.50 and Rs. 4.00 resp. The average productivity per acre of potatoes is 4 tonnes, carrots are 6 tonnes and 2 tonnes of beans is produced per acre. Fertilizer is available at rate of Rs. 1.0 per kg. and amount of fertilizer required for potatoes and carrots is 200 kg per acre and 100 kg for beans. Labour required for sowing, cultivating and harvesting per acre is 5 man – days for potatoes and carrots, and 6 man-days for beans. A total of 400 man-days of labour are available at Rs. 60 per man day.

Formulation of the linear programming to maximize farmer's total profit.

Sol. How much area should be given to each type of crop for cultivation so that profit from the production can be maximized, it is to be decided.

Let $x_1, x_2, \&\ x_3$ be the acres of land for different crops (potatoes, carrots, and beans resp.). Therefore the crops production will be $4000x_1$ of potatoes, $6000x_2$ of carrots & $2000x_3$ of beans

Total sales = $[2 \times 4000x1 + 1.5 \times 6000x2 + 4 \times 2000x3] = 8000x_1 + 9000x_2 + 8000x_3$

Total expenses on fertilizers = $[1 \times \{200x(x1 + x2) + 100x3\}] = 200x_1 + 200x_2 + 100x_3$

Labour expenses = $[60 \times \{5x(x1 + x2) + 6x3\}] = 300x_1 + 300x_2 + 360x_3$

Net profit (P) = Total sales – Total expenses

$= 8000x_1 + 9000x_2 + 8000x_3 - 500x_1 - 500x_2 - 460x_3 = 7500x_1 + 8500x_2 + 7540x_3$

Since total area is restricted to 100 acres of land, $x_1 + x_2 + x_3 \leq 100$

Also, the total man-days labour is restricted to 400 man – days, therefore

$5x_1 + 5x_2 + 6x_3 \leq 400$

Hence the farmer's allocation problem can be finally put in the form

Find the values of x_1, x_2, x_3 so as to maximize
$P = 7500x_1 + 8500x_2 + 7540x_3,$
subject to the conditions
$x_1 + x_2 + x_3 \leq 100$
$5x_1 + 5x_2 + 6x_3 \leq 400$
Where each $x_1, x_2, x_3 \geq 0$

9.7 Few Important Definitions

There are few definitions which may help in understanding of forthcoming topics

i. Solution to LPP: Any set of the type $X = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ of variables is called a solution to linear programming problem if it satisfy the following constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ldots + x_{n+m} = b_m$$

ii. Feasible solution: Any set of the type $X = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ of variables is called a solution to linear programming problem if it satisfy the above (in i) constraints and non negativity restrictions also mentioned below

where $x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0, x_{n+1} \geq 0 \ldots x_{n+m} \geq 0$

iii. Basic Solutions: A basic solution to the set of constraints (in i) is a solution obtained by setting any n variables (among m+n variables0 equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables (of course, some of them can be zero) are called basic variables and remaining n non zero-valued variables are called non-basic variables.

The no. of basic solutions obtained will be the most ${}^{m+n}C_m = (m+n)! / n!\, m!$, which is the number of combinations of n+m things taken m at a time.

iv. Basic Feasible Solution: A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions $x_1 \geq 0, x_2 \geq 0, \ldots\ldots x_n \geq 0, x_{n+1} \geq 0, \ldots\ldots x_{n+m} \geq 0$ i.e. all basic variables are non-negative. These are of two types

a. Non-degenerate BFS: A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i (i= 1,2,3........,m). In other words, all m basic variables are positive and the remaining n variables will be all zero.

b. Degenerate BFS: A basic feasible solution is called degenerate, if one or more basic variables are zero-valued.

v. Optimum Basic Feasible Solution : A basic feasible solution is said to be optimum if it also optimizes (maximizes or minimizes) the objective function

Max. $z = c_1 x_1 + c_2 x_{2+\ldots\ldots\ldots\ldots} + c_n x_n + 0\, x_{n+1\ldots\ldots} + 0\, x_{n+m}$

Max. $z = c_1 x_1 + c_2 x_{2+\ldots\ldots\ldots\ldots} + c_n x_n + 0\, x_{n+1\ldots\ldots} + 0\, x_{n+m}$

vi. Unbounded solution: If the value of the objective function z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

9.8 Applications

In the present world, most of the events are non-linear type, yet some of the events occurring in routine life are linear type or can be so approximated. Therefore, linear programming and its application must be understood by the managers in solving the problems.

LP programming is widely used in various fields such as military, businesses, industries, marketing, distribution and advertising problems to provide solutions. Reasons for its wide usage are

1. In L.P. models, a lot of varying data can be handled very easily.
2. A lot of powerful and efficient techniques are available to solve problems.
3. From various fields, problems can be easily represented through linear programming problems.

L.P. needs a lot of data to be handled which is complex and elaborative, in present scenario of computer world, this disadvantage has been overcome completely as these computers can handle even large no. of variables in L.P. problems in a short duration which can take days, months and even years if it is done manually.

Application of Linear Programming in different areas are given below

Environment Protection

A lot of pollution in the form of carbon particles in air, waste material in water resources is taking place. L.P. helps in analysis of different alternatives for efficient waste disposal, recycling of the material and energy saving and utilising policies.

Transportation Problems

We can understand by taking an example, x no. of products / equipments / parts to be transported to y no. of destinations. Cost of transportation to each destination, demand of destinations and storage costs of each destination are known. The problem is to design the most appropriate transportation plan with schedule and quantities which minimizes the total transportation cost.

Assignment Problems

These are used to solve the problems when there are so many jobs to be performed and there are lot of alternatives to complete the job at different stages (with availability of resources are limited). Assignment model is that in which no. of allocation of jobs is same as that of no. of resources (men). The assignment problem becomes more complex when some of the jobs require more than one resource or if the resource can be used for more than one job. Time required to complete the job at each source is known. Now, optimum solution is to be found out which gives total minimum time to perform all the jobs.

Diet problems

It is another important category where linear programming is applied. Nutrient contents such as fats, carbohydrates, vitamins, proteins, starch etc. is known in various food products. Minimum daily requirement of each nutrient in the diet as well as the cost of each type of food stuff is given and the problem is to determine the minimum cost diet which will satisfy the minimum nutrients requirement on daily basis.

Manpower scheduling problems

These types of problems are occurring in a lot of areas like big industries, hotels, hospitals, ports, etc. To utilise the manpower resources in the most optimum manner, LP. Techniques are used which helps in reducing the overtime costs (& other associated costs like transportation, wages etc. by allocating right no. of manpower in different shifts (day, night etc.).

Agricultural problems

It is a very big area where there are lot of resources are used like seeds, machines for harvesting, fertilisers, labour, land and capital to invest. The concern is to maximise the revenues and minimising costs.

9.9 Assumptions

Following are the assumptions for linear programming problem.

Additivity: To understand this,

if a machine takes **t1** time to produce one unit of item P1

and **t2** time to produce one unit of item P2,

then time required to produce one item each of P1 & P2 is **t1+t2**.

Proportionality: It is the primary requirement of linear programming problem that the objective function and every constraint function must be linear. In simple words, if 1 kg of any item costs Rs. 5, then cost of 20 kg of the item will be 100 Rs. If the capacity of any steel plant is 1000 MT per day, it can produce 25000MT (25 x 1000) in one month (for 25 days working).

Multiplicativity: For producing one item on machine, it takes 2 hrs.,10 items of similar type will be produced in 20 hrs.

Divisibility: It means that the fractional levels of variables must be permissible besides integral values. This also means the variables are assumed to be continuous.

Deterministic: All the parameters in the linear programming models are assumed to be known exactly. While in actual practice, production may depend upon chance also.

9.10 Solution of LP

Once a problem is formulated as mathematical model the next step is to solve the problem to get the optimal solution. A linear programming problem with two variables can be solved by graphical method. The procedure of graphical method can be understood as follows

i. Each inequality constraint should be considered as equation.

ii. Each equation must be plotted on the graph and each equation will be geometrically represented by a straight line.

iii. Feasible region will be shaded. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to the line is '$\leq$ ' than the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint with ' $\geq$" ' sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the 'feasible region'.

iv. The convenient value of z will be chosen (say = 0) and the objective function line can be plotted.

v. The objective function line will be pulled until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

vi. The coordinates of the extreme points can be read selected in step (v) and find the maximum or minimum value of z.

Graphical Procedure can be understood by the following examples

Example 1 Find a geometrical interpretation and solutions as well for the following LP problem.

Max. $z = 3a_1 + 5a_2$ subject to restrictions
$a_1 + 2a_2 \leq 2000$,
$a_1 + a_2 \leq 1500$,
$a_2 \leq 600$ and
$a_1, a_2 \geq 0$
Graphical Solution

Step i We will consider two mutually perpendicular lines ox_1 and ox_2 as axes of coordinates. Obviously, any point (a_1, a_2) in the positive quadrant will certainly satisfy non-negativity restrictions $a_1, a_2 \geq 0$. To plot the line $a_1 + 2a_2 = 2000$, put $a_2 = 0$, find $a_1 = 2000$ from this equation.

We will mark a point L such that OL = 2000 by assuming a suitable scale, say 500 units = 2cm. Similarly, when we put $a_1 = 0$ to find $a_2 = 1000$ and mark another point M such that OM = 1000. Now we will join the points L & M. The line will represent the equation $a_1 + 2a_2 = 2000$ as shown in the figure

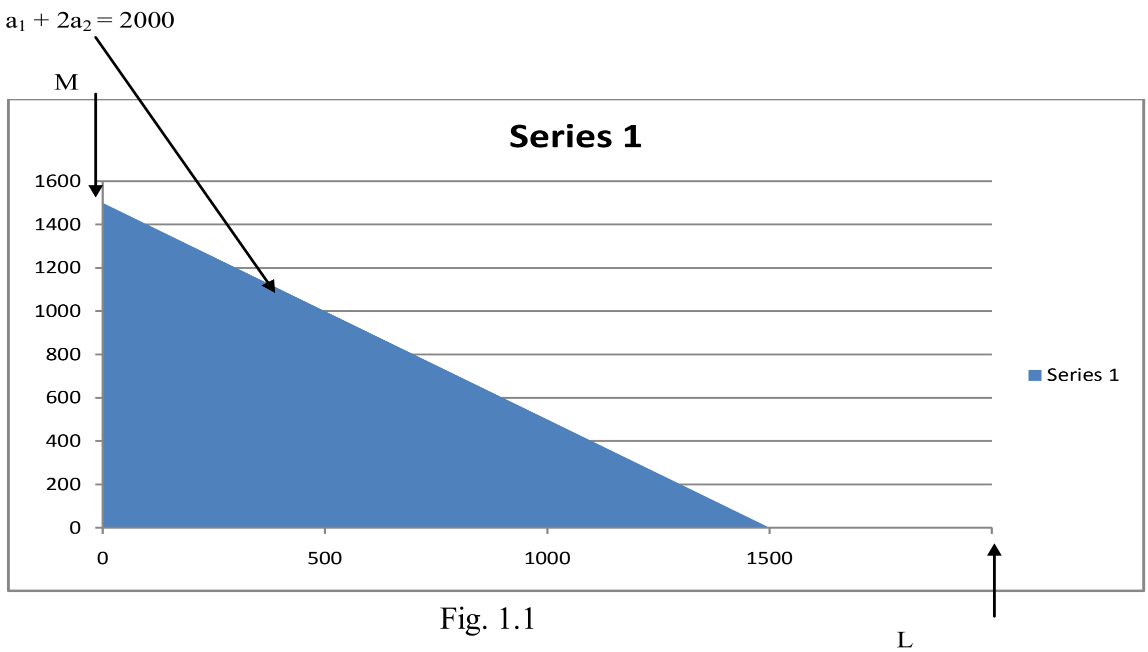


Fig. 1.1

Point P lying on or below the line $a_1 + 2a_2 = 2000$ will satisfy the inequality $a_1 + 2a_2 \leq 2000$ (if we take a point (500, 500), i.e. $a_1 = 500$, $a_2 = 500$, then putting the values in the equation, we have $500 + 2 \times 500 \leq 2000$, which is true)

Similar procedure is now adopted to plot the other two lines

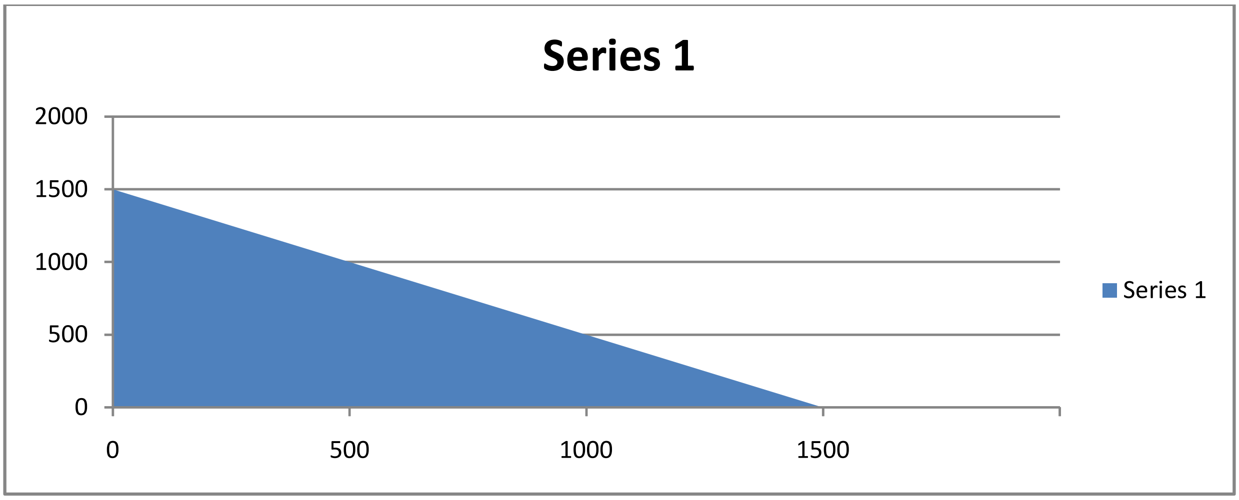


Fig. 1.2

and $a_2 = 600$

Any point on or below the lines $a_1 + a_2 = 1500$ and $a_2 = 600$ will satisfy the given.

Inequalities

$a_1 + a_2 \leq 1500$

and $a_2 \leq 600$ respectively.

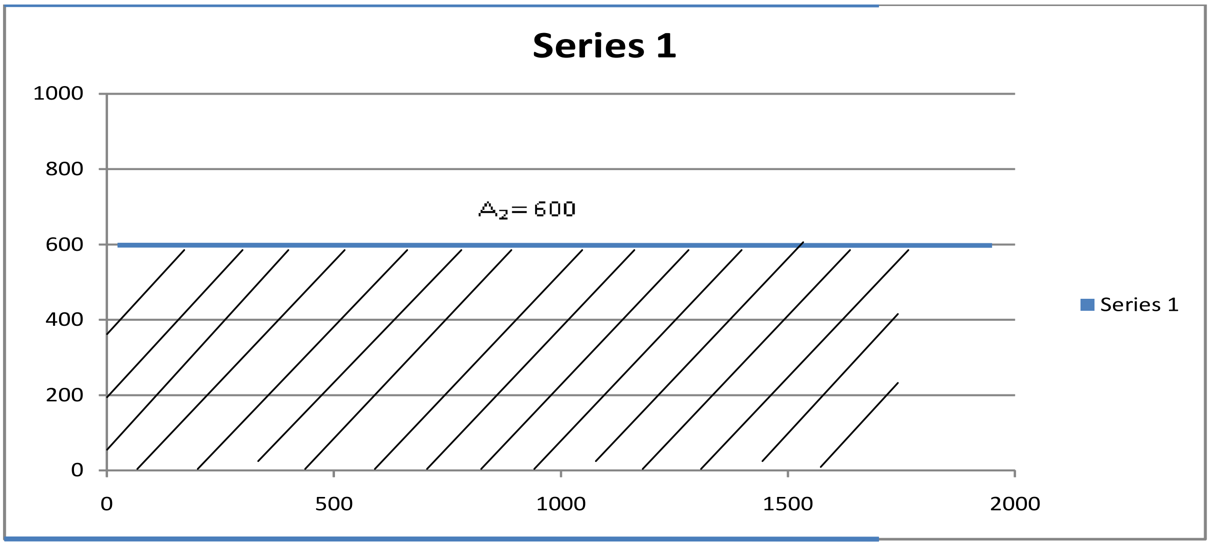
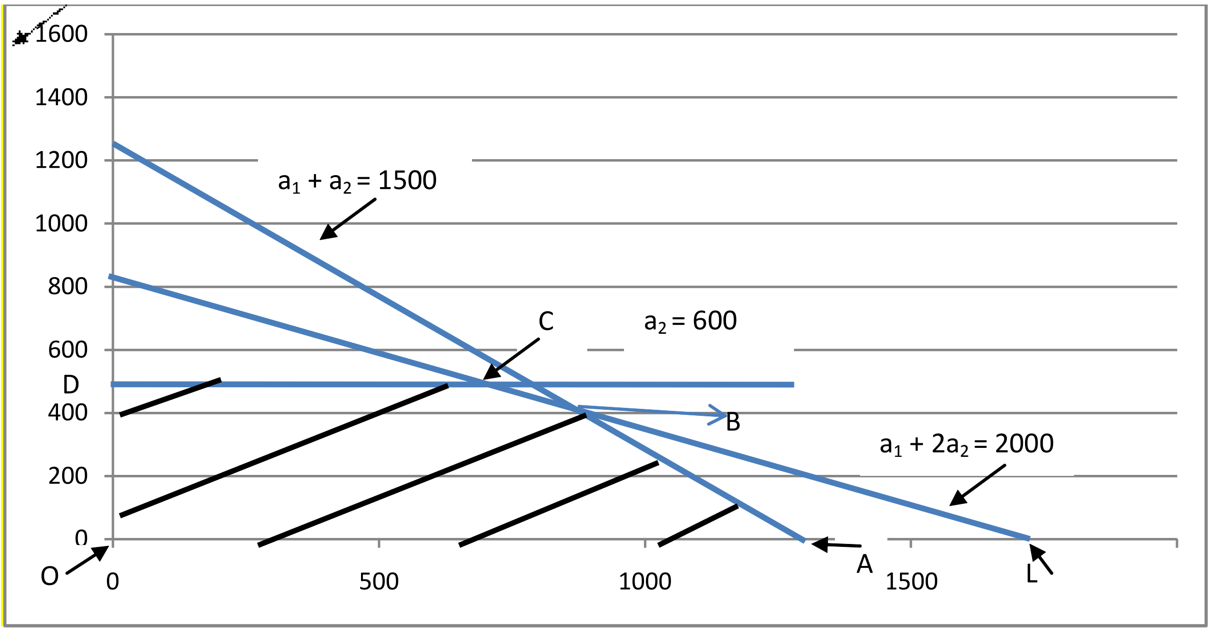


Fig. 1.3

Step II We will find the feasible region or solution space by combining the figures 1.1, 1.2 and 1.3. A common shaded area OABCD can be obtained which is a set of points satisfying the inequality constraints

$a_1 + 2a_2 \leq 2000$ and $a_1 + a_2 \leq 1500$, $a_2 \leq 600$ and non-negativity restrictions as $a_1, a_2 \geq 0$

Hence any point in the shaded area (including the boundary) is feasible solution to given LPP.



Step III Co-ordinates of the corner points of feasible region O, A, B, C, D can be found out.

Step IV The corner points of optimal solution can be located either by calculating the value of z for each corner point O, A, B, C and D (or by adopting the following procedure)

Here, The problem is to find out the point or points in the feasible region which maximize the objective function. For some fixed value of z, $z = 3a_1 + 5a_2$ is a straight line and any point on it gives the same value of z. Also, it should be noted that the lines corresponding to different values of z are parallel because the gradient (-3/5) of the line $z = 3a_1 + 5a_2$ remains the same throughout. For z = 0, i.e. $0 = 3a_1 + 5a_2$, means a line which passes through the origin. To draw the line $3a_1 + 5a_2 = 0$ determine the ratio $a_1 / a_2 = -5 / 3 = -500 / 300$

We can make a point E moving 500 units distance from the origin on the negative side of a_1 – axis. We may find the points F such that EF = 300 units in the positive direction of a_2 axis. Joining the point F & O, we can draw the line $3a_1 + 5a_2 = 0$. We draw the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of z is attained. It is also possible that such a line may coincide with one of the edge of feasible region. In that case, every point on the edge gives the maximum value of z.

In this example, maximum value of z is attained at the corner point B (1000, 500) which is the point of intersection of lines $a_1 + 3a_2 = 2000$ and $a_1 + a_2 = 1500$. Hence the required solution is $a_1 = 1000$, $a_2 = 500$ and max. value of z = Rs. 5500

Simplex Method is an iterative procedure for solving linear programming problem. It consists of the following steps

a) Having a trial basic feasible solution to constraint-equations

b) Testing whether it is an optimal solution

c) Improving the first trial solution by a set of rules and repeating the process till an optimal solution is obtained.

The procedure requires at most m(= no. of equations) non-zero variables in the solution. One of the non-basic variables at one iteration becomes basic at the following iteration is called an entering variable and one of the basic variables at one iteration becomes non-basic at the following iteration and is called a departing variable.

For convenience, construct the LP problem in standard form

Max. $z = c_1x_1 + c_2x_2 + c_3x_3 + \text{-----------------} + c_nx_n + 0x_{n+1} + 0x_{n+2} + \text{-------------} + 0x_{n+m}$

S.T.

$a_{11}x_1 + a_{12}x_2 + \text{----------------------} + a_{1n}x_n + x_{n+1} \qquad\qquad\qquad = b_1$

$a_{21}x_1 + a_{22}x_2 + \text{----------------------} + a_{2n}x_n \qquad\qquad + x_{n+2} \qquad = b_2$

$a_{m1}x_1 + a_{m2}x_2 + \text{----------------------} + a_{mn}x_n \qquad\qquad + x_{n+m} \qquad = b_m$

and $x_1 \geq 0$, $x_2 \geq 0$,------- , $x_n \geq 0$, $x_{n+1} \geq 0$ ------ , $x_{n+m} \geq 0$

Step 1. Basic feasible solution is $x_1 = x_2 = x_3 = \text{-------} = x_n = 0$ (non-basic variables)

$x_{n+1} = b_1$, $x_{n+2} = b_2$, ------------ , $x_{n+m} = b_m$, (basic variables)

Step 2. Now construct the starting simplex table

			c_j	c_1	c_2	c_n	0	0	0	
Basic Variables	C_B	X_B	x_1	x_2 ------------	x_n		x_{n+1}	x_{n+2} -----------	x_{n+m}	MIN RATIO
x_{n+1}	0	b_1	a_{11}	a_{12}	a_{1n}		1	0	0	
x_{n+2}	0	b_2	a_{21}	a_{22}	a_{2n}		0	1	0	
x_{n+m}	0	b_m	a_{m1}	a_{m2}	a_{mn}		0	0	1	
	$z = C_BX_B$		Δ_1	Δ_2	Δ_n		0	0	0	$\Delta_j = C_BX_j - c_j$

Step 3. Incoming vector is considered corresponding to most –ve value of Δ_j i.e. Δ_k

Step 4. Optimality test

a) If all $\Delta_j \geq 0$, the solution under test will be optimal.

b) If atleast one Δ_j is negative , the solution under test is not optimal then proceed to improve the solution in the next step.

c) If corresponding to any negative Δ_j, all elements of the column $X_j \leq 0$ then the solution under test will be unbounded.

Step 5. Compute min ratio X_B / x_k for $x_k > 0$

Step 6. Outgoing vector is minimum of min ratio.

Step 7. Element at the intersection of minimum min ratio and incoming vector is key element. Now divide all the elements of row by key element and make other elements of key column zero. Then we get revised simplex table and go to step 4.

Example 1. Solve by simplex method

Min. $-z = a_1 - 3a_2 + 2a_3$ subject to

$3a_1 - a_2 + 3a_3 \leq 7$,

$-2a_1 + 4a_2 \leq 12$,

$-4a_1 + 3a_2 + 8a_3 \leq 10$

$a_1, a_2, a_3 \geq 0$

Sol. The problem is of minimization. Converting the objective function from minimization to maximization, we have

Max. $-z = -a_1 + 3a_2 - 2a_3$ = Max. z', where $-z = z'$

$3a_1 - a_2 + 3a_3 + a_4 = 7$,

$-2a_1 + 4a_2 + a_5 = 12$,

$-4a_1 + 3a_2 + 8a_3 + a_6 = 10$

Basic Variables	**C_B**	**X_B**	**a_1**	**a_2**	**a_3**	**a_4**	**a_5**	**a_6**	**Min. Ratio (X_B / a_k)**
a_4	0	7	3	-1	3	1	0	0	---
a_5	0	12	-2	4	0	0	1	0	12/4
a_6	0	10	-4	3	8	0	0	1	10/3
$a_1 = a_2 = a_3 = 0$	$z' = 0$, $z = 0$		1	-3*	2	0	0	0	← Δ_j
a_4	0	10	5/2	0	3	1	¼	0	10 / 5/2
a_2	3	3	-1/2	1	0	0	¼	0	---
a_6	0	1	-5/2	0	8	0	-3/4	1	---
$a_1 = a_2 = a_3 = 0$	$Z' = 9$, $\therefore z = -9$		-1/2*	0	2	0	¾	0	← Δ_j
a_1	-1	4	1	0	6/5	2/5	1/10	0	
a_2	3	5	0	1	3/5	1/5	3/10	0	
a_6	0	11	0	0	11	1	-1/2	1	
$a_3 = a_4 = a_2 = 0$	$z' = 11$, $z = -11$		0	0	13/5	1/5	8/10	0	$\Delta_j \geq 0$

The optimal solution is $x_{1=4}$, $x_2 = 5$, $x_3 = 0$, Min $z = -11$

9.10.1 Limitations

There are some limitations also with linear programming in spite of wide usage

i. Parameters used in the model are assumed to be constant, but in real life situation, they are neither constant nor deterministic.

ii. There are some problems in real life situation where objective functions and constraints are not linear. Concerning business and industrial problems, constraints are not treated linearly.

iii. Integer valued solutions are not guaranteed. For example, if calculation of men & machine is required to do some particular job, rounding off the solution to the nearest integer will not give an optimal solution.

iv. Sometimes it is not possible to sort out large scale problems in linear programming in spite of availability of computers. The problem can be solved by dividing it into lot of small problems and then sorting out those problems.

v. In Linear programming problems, effect of time and uncertainty are not taken into consideration.

vi. Linear programming deals with only single objective, whereas in real life situations, problems are having multiple objectives.

9.10.2 Advantages

Following are the advantages of linear programming problem.

i. The losses occurring due to bottlenecks can be reduced effectively by implementation of linear programming technique. In production processes, highlighting bottlenecks is one of the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet demand while others remain idle for some times.

ii. Helpful in decision making. By the use of decision making techniques, the user becomes more objective and less subjective.

iii. Whenever applicable, linear programming technique helps us in making the optimum utilization of production resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.

iv. Linear programming techniques provide practically applicable solutions since there may be other constraints operating outside the problem which must also be taken into consideration.

9.11 Transportation Problems

It is a process of transportation of various amounts of a single homogeneous commodity which are initially stored at various locations to different destinations in such a manner that cost of transportation is minimized. e.g., a pharmaceutical company which has lot of plants at different locations. The company has lot of retail dealers at different locations in different cities of the country. By the use of transportation model / technique, transportation schedules can be made which can reduce the cost of transportation to minimum.

9.11.1 Mathematical Formulation

Mathematical Formulation can be explained as

Let us take few assumptions m – origins (production unit), i^{th} origin processes a_i units of a certain products, n no. of destinations (which may or may not be equal to m) with destination j requiring b_j units. Cost of transportation from different sources (production units) to different destinations (retail dealers), directly or indirectly can be calculated depending upon distances, shipping hours etc. c_{ij} be the cost of shipping x_{ij} unit product from i^{th} origin j^{th} destination.

We also assume that total availabilities $\sum a_i$ at production units satisfy the total requirements $\sum b_j$ at destinations i.e.

$$\sum a_i = \sum b_j \quad (i = 1, 2, 3, ..., m; j = 1, 2, ..., n)$$

(In any case, where $\sum a_i \neq \sum b_j$, manipulation can be done to make it to $\sum a_i = \sum b_j$)

Now, the problem is to find out, non-negative ($\geq$) values of x_{ij} satisfying both, the availability constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{for } (i = 1, 2, 3, \ldots\ldots, m;) \qquad \ldots\ldots\ldots\ldots(A)$$

as well as the requirement constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } (j = 1, 2, \ldots\ldots, n) \qquad \ldots\ldots\ldots\ldots(B)$$

and minimizing the total cost of transportation

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} \quad \text{(objective function)} \qquad \ldots\ldots\ldots\ldots(C)$$

It may be seen from the equations that constraints equations (A), (B) & the objective function (C) are all linear in x_{ij} so it may be considered as linear programming problem.

9.11.2 Matrix Form

If we take, the transportation problem represented as mathematically formulated above, the set of constraints constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1, 2, 3, \ldots\ldots, m;)$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } (j = 1, 2, \ldots\ldots, n)$$

represent m+n equations in mn non-negative variables $x_{ij.}$ Each variable x_{ij} appears exactly in two constraints, one associated with the ith origin Oi and other with jth destination Dj. In the above ordering of constraints, first we write the origin equations and then destination equations. Then the transportation problems can be stated in matrix form as follows

Minimize $z = CX$, $X \varepsilon R^{mn}$, subject to the constraints $AX = b$, $x \geq 0$, $b \varepsilon R^{m+n}$

Where $x = [x_{11,} x_{12\ldots} x_{1n,} x_{21,\ldots\ldots}, x_{2n,\ldots\ldots} x_{m1,\ldots\ldots}, x_{mn.}]$, C is the cost factor, $b = [a_{1,} a_{2,\ldots\ldots} a_m, b_{1,} b_{2,\ldots\ldots} b_n)$ and A is an (m+n) x mn real matrix containing the coefficient of constraints.

It is worth noting that elements of A are either 0 or 1. Thus the general LPP can be reduced to transportation problem if

(i) a_{ij} 's are restricted to values zero and one and(0 & 1);
(ii) The units among the constraints are homogeneous.

For e.g., if m=2 and n=3, the matrix A is given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{(1)}_{23} & e^{(1)}_{23} \\ I_3 & I_3 \end{pmatrix}$$

Therefore, for a general transportation problem, we may write

$$A = \begin{matrix} e^{(1)}_{mn} & e^{(2)}_{mn} & \ldots\ldots & e^{(m)}_{mn} \\ I_n & I_n & \ldots\ldots & I_n \end{matrix}$$

Where $e^{(i)}_{mn}$ is an m x n matrix having a row of unit elements as its ith row and 0's everywhere else and I_n is the n x n matrix

If a_{ij} denotes column vector of A associated with any variable x_{ij}, then it can be easily verified that

$a_{ij} = e_i + e_{m+j}$ where $e_i, e_{m+j} \varepsilon$ R are unit vectors

9.11.3 Few Definitions

The terms feasible solution, basic feasible solution and optimum solution may be formally defined with reference to the transportation problem (T.P.) as follows

i. Feasible Solution: It is a set of non-negative individual allocations (x_{ij}) which simultaneously removes deficiencies, called as feasible solution.

ii. Basic Feasible Solution : A feasible solution to m-origin, n-destinations problem is basic solution if the no. of positive allocations are m+n-1 i.e., one less than the sum of rows and columns (which satisfies the non-negativity restrictions $x_1 \geq 0, x_2 \geq 0, \ldots . x_n \geq 0, x_{n+1} \geq 0 \ldots . x_{n+m-1} \geq 0$

a. Degenerate BFS: A non-degenerate basic feasible solution is the basic feasible solution which has less than m+n-1 positive allocations x_i (i.e. i< 1,2,3........,m+n-1).

b. Non-Degenerate: BFS A basic feasible solution is called degenerate, if condition referred in Degenerate BFS is not satisfied.

iii. Optimum Basic Feasible Solution: A basic feasible solution is said to be optimum if it minimizes the total transportation cost.

9.12 Tabular Representation

Let us suppose that there are m factories and n warehouses, the transportation problem can be shown in a tabular form as below

Warehouse→ **Factory↓**	W_1	W_2	......	W_j	....	W_n	Factory capacities
F_1	c_{11}	c_{12}	......	c_{1j}	......	c_{1n}	a_1
F_2	c_{21}	c_{22}	......	c_{2j}	......	c_{2n}	a_2
			......		......		
F_i	c_{i1}	c_{i2}	......	c_{ij}	......	c_{in}	a_i
			......		......		
F_m	c_{m1}	c_{m2}	......	c_{mj}	......	c_{mn}	a_m
Warehouse requirements	b_1	b_2	......	B_j	......	B_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Warehouse → Factory ↓	W_1	W_2	……	W_j	…….	W_n	**Factory capacities**
F_1	x_{11}	x_{12}	……	x_{1j}	……	x_{1n}	a_1
F_2	x_{21}	x_{22}	……	x_{2j}	……	x_{2n}	a_2
			……		……		
F_i	x_{i1}	x_{i2}	……	x_{ij}	……	x_{in}	a_i
			……		……		
F_m	x_{m1}	X_{m2}	……	x_{mj}	……	x_{mn}	a_m
Warehouse requirements	b_1	b_2	……	b_2	……	b_2	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Two table shown above can be combined by inserting each unit of product cost c_{ij} with the corresponding amount x_{ij} into the cell (i, j).

The product of c_{ij} & (x_{ij}) gives us the net cost of transporting x_{ij} units from Factory F_i to Warehouse W_j.

9.13 Solution of Transportation Problem

There are several methods for solution of Transportation Problem given below

North-west corner method

The method may be stated as follows

a. Start from the upper left corner of the requirements table i.e. the transportation matrix framed in step I and compare the supply of plant I (call it S_1) with the requirement of distribution centre 1 (call it D_1)

   i. If $D_1 < S_1$, if the amount required at D_1 is less than the number of units available at S_1, set x_{11} equal to D_1, find the balance supply and demand and proceed to cell (1,2) (i.e., proceed horizontally)

Distribution centres

Plants		1	2	3	4	Supply
	1	2 (6)	3	11	7	**6/0**
	2	1 (1)	0 (0)	6	1	**1/0**
	3	5	8 (5)	15 (3)	9 (2)	**10/5/2/0**
Requirement		7/1/0	5/0	3/0	2/0	

   i. If $D_1 = S_1$, set x_{11} equal to D_1, compute the balance supply and demand and proceed to cell (2,2) (i.e. proceed diagonally), also make a zero allocation to the least cost cell in S_1 / D_1

   ii. If $D_1 > S_1$, set x_{11} equal to S_1, compute the balance supply and demand and proceed to cell (2,1) (i.e. proceed vertically).

b. We continue in this manner, step by step, away from the north-west corner until, finally, a value is reached in the south-east corner.

We proceed as follows in this particular example

i. Set x_{11}, equal to 6, namely, the smaller of the amounts available at S_1 (6) and that needed at D_1 (7) and

ii. We proceed to cell (2, 1) (rule c). Compare the number of units available at S_2 (namely 1) with the amount required at D_1 (1) and accordingly set x_{21} = 1. Also set x_{22} = as per rule (b) above.

iii. We proceed to cell (3,2) (rule b). Now supply from plant S_3 is 10 units while the demand for D_2 is 5 units; accordingly set x_{32} equal to 5.

iv. We proceed to cell (3,3) (rule a) and allocate 3 there.

v. We proceed to cell (3,4) (rule a) and allocate 2 there.

It can be easily seen that the proposal solution is a feasible solution since all supply and demand constraints are fully satisfied.

The following points may be noted in connection with this method

i. The quantities allocated are put in parenthesis and they represent the values of the corresponding decision variables. These cells are called basic or allocated or occupied or loaded cells. Cells without allocations are called non-basic or vacant or empty or unoccupied or unloaded cells. Values of the corresponding variables are all zero in these cells.

ii. This method of allocation does not take into account the transportation cost and, therefore, may not yield a good (most economical) initial solution. The transportation cost associated with this solution is

Z = Rs. [2 x 6 + 1 x 1 + 8 x 5 + 15 x 3 + 9 x 2] x 100 = Rs. 11,600

Vogel's Approximation Method (VAM)

In the transportation matrix if an allocation is made in the second lowest cost cell instead of the lowest, then this allocation will have associated with it a penalty corresponding to the difference of these two costs due to 'loss of advantage'. That is to say, if we compute the difference between the two lowest costs for each row and column, we find the opportunity cost relevant to each row and column. It would be most economical to make allocation against the row or column with the highest opportunity cost. For a given row or column, the allocation should obviously be made in the least cost cell of that row or column.

Substep I Write down the cost matrix as below

Distribution centres

		1	2	3	4	Supply
	1	2	3	11	7	**6[1]**
Plants	2	1	0	6	1 (1)	**1/0[1]**
	3	5	8	15 (3)	9 (2)	**10[3]**
Requirement		7	5	3	2	
		[1]	[3]	[5]	[6]	

Enter the difference between the smallest and second smallest element in each column below the corresponding column and the difference between the smallest and second smallest

Element in each row to the right of the row. Put these numbers in brackets as shown. For example in column 1, the two lowest elements are 1 & 2 and their difference is 1 which is entered as [1] below column 1. Similarly, the two smallest elements in row 2 are 0 and 1 and their difference 1 is entered as [1] to the right of row 2. A row or column "difference" can be thought of a penalty for making allocation in second smallest cost cell instead of smallest cost cell. In other words this difference indicates the unit penalty incurred by failing to make an allocation to the smallest cost cell in that row or column. In case the smallest and the second smallest elements in a row / column are equal , the penalty should be taken as zero.

Substep 2 Select the row or column with the greatest difference and allocate as much as possible within the restriction of the conditions to the lowest cost cell in the row or column selected.

In case of tie among the highest penalties, select the row or column having minimum cost. In case of tie in the minimum cost also, select the cell which can have maximum allocation. If there is tie among maximum allocation cells also, select the cell arbitrarily for allocation. Following these rules yields the best possible initial basic feasible solution and reduce the number of iterations required to reach the optimal solution.

Thus since [6] is the greatest number in brackets, we choose column 4 and allocate as much as possible to the cell (2,4) as it has the lowest cost 1 in column 4. Since supply is 1 while the requirement is 2, maximum allocation is (1).

Substep III Cross out of the row or column completely satisfied by the allocation just made. For the assignment just made at (2,4), supply of plant 2 is completely satisfied. So, row 2 is crossed out and the shrunken matrix is written below

		1	2	3	4
1	2	3 (5)	11	7	6/1[1]
3	5	8	15	9	10[3]
		7	5/0	3	1
		[3]	[5]	[4]	[2]

This matrix consists of the rows and columns where allocations have not yet been made including revised row and column totals which take the already made allocation into account.

Substep IV Repeat steps 1 to 3 until assignments have been made.

a) Column 2 exhibits the greatest difference of [5]. Therefore, we allocate (5) units to cell (1,2) , since it has the smallest transportation cost in column 2. Since requirements of column 2 are completely satisfied, this column is crossed out and the reduced matrix is written against as the table below

Supply

		1	3	4	
Plants	1	2 (1)	11	7	1/0 [5]
	3	5	15	9	10[4]
		7/6	3	1	
		[3]	[4]	[2]	

b) Differences are recalculated. The maximum difference is [5]. Therefore, we allocate (1) to the cell (1,1) since it has the lowest cost in row 1. Since requirements of row 1 are fully satisfied, it is crossed out and reduced matrix is given below.

In table below, it is possible to find row differences but it is not possible to find column differences. Therefore, remaining allocations in this table are made by following the least cost method.

		1	3	4	
Plants	3	5	15	9	10/4/3/0
		6/0	3/0	1/0	

c) As cell (3,1) has the lowest cost 5, maximum possible allocation of (6) is made here. Likewise, next allocation of (1) is made in cell (3,4) and (3) in cell (3,3) as shown

All allocations made during the above procedure are shown below in the allocation matrix.

Distribution centres

		1	2	3	4	Supply	
Plants	1	2 (1)	3 (5)	11	7	6/1/0	[1][1][5]
	2	1	0	6	1 (1)	1/0	[1]
	3	5 (6)	8	15 (3)	9 (1)	10/4/3/0	[3][3][4]
		7	5	3	2/1		
		[1]	[3]	[5]	[6]		
		[3]	[5]	[4]	[2]		
		[3]		[4]	[2]		

The cost of transportation associated with the above solution is

Z = Rs. (2 x 1 + 3 x 5 + 1 x 1 + 5 x 6 + 15 x 3 + 9 x 1) x 100

= Rs. (3 + 15 + 1 + 30 + 45 + 9) x 100 = Rs. 10, 200

Which is evidently least of all the value of transportation cost found by different methods? Since Vogel's approximation method results in the most economical initial feasible solution, we shall use this method for finding such a solution for all transportation problems henceforth.

9.14 Summary

This unit provides you a detailed description about linear programming problem and transportation problem, its assumptions, advantages and disadvantages. it also describes various methods for solving linear programming problem and transportation problem.

Linear programming is a major innovation since World War II in the field of business decision making under condition of certainty. Linear means that the relationships are represented by straight lines and the programming means taking decisions systematically. Thus Linear programming is a decision making technique under given constraints on the assumptions that the relationships amongst the variables representing different phenomenon happen to be linear. We describe two methods for solving linear programming problem i.e. Graphical method and Simplex method. Graphical method is applicable only to problems having two variables. But Simplex method is used for problems having two variables as well as more than two variables. Some linear programming problems involving several product sources and several destinations of products. Such problems are known as Transportation problems. A common feature of such problems is that all the units available must be assigned. The method applicable to such problems is known as Transportation method. This method is used to optimize the distribution costs.

9.15 Key Words

- **Linear Programming:** A mathematical technique for finding the optimal uses of an organization resources. In mathematical expression, Equation of the form defined as $a_1\ a_{1+}\ a_2\ a_{2+ \ldots\ldots\ldots +}\ a_n\ a_{n.}$ is LP.
- **Objective:** The ultimate goal of the organization.
- **Inequality:** Mathematical expression indicating that minimum/maximum requirements must be met..
- **Constraint:** A limitation on the availability of resources.
- **Slack variables:** We have to add something positive to the left hand side in order to make it an equality if a constraint has $\leq$ sign.
- **Surplus variables:** We have to subtract something positive to the left hand side in order to make it an equality if a constraint has $\geq$ sign.
- **Objective Function:** Expression showing the relationship between the variables in the given problem and the organization.
- **Feasible Region:** The area containing all the possible solutions which satisfy all the constraints in the problems.
- **Infeasibility:** The condition when there is no solution which satisfy all the constraints in the problem.
- **Simplex Method:** A systematic and an efficient method for solving a LPP.

9.16 Self Assessment Test

1. What is Linear Programming? Explain applications in various areas.
2. Explain different assumptions and limitations in LP?
3. Explain
    - General formulation of LP
    - Standard formulation of LP
    - Matrix formulation of LP
4. XYZ foods company is developing a low calories high protein diet supplement called Hi-Pro. The specification have been established by a panel of experts.These specifications given in table

Units of Nutritional Elements per gm Serving of Basic Foods

Nutritional Elements	Basic Foods			Hi-Pro. Specifications
	1	2	3	
Calories	300	200	150	≤ 400
Protein	200	150	100	≥ 300
Vitamin B	125	200	140	≥ 200
Vitamin A	50	135	230	≥ 150

Formulate the LPP to minimize cost.

5. Solve the following LPP by graphical method & Simplex method

a) Max $z = 3x_1 + 2x_2$ s.t. $-2x_1 + 3x_2 \leq 9$, $x_1 - 5x_2 \geq -20$; $x_1, x_2 \geq 0$

[Unbounded solution]

b) Max. $z = 120x + 100y$ s.t. $10x + 5y \leq 80$, $6x + 6y \leq 66$, $4x + 8y \geq 24$,

$5x + 6y \leq 90$; $x, y \geq 0$

[x= 500 , y = 600 , Max. z = 1200]

c) Min. $z = 5x + 3y$ s.t. $x + y \leq 2$, $5x + 2y \leq 10$, $3x + 8y \leq 12$; $x, y \geq 0$

[x = 2 , y = 0 , z = 10]

9.17 References

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Unit - 10 : Networks

Unit Structure:

10.0 Objectives

After studying this unit, you should be able to understand

- Meaning of Project
- Characteristics of a project
- Project Management Phases
- CPM and PERT development
- Presentation of network diagram
- Rules regarding presentation of network diagram
- Critical Path Method for computation of scheduling
- PERT Method and its advantages and disadvantages

10.1 Introduction

A project is defined as a set of inter related activities which should be completed in a certain order to complete the entire project. These activities may or may not be dependent on each other and these have certain logical sequence as one cannot be started unless some other activities are completed. An activity in project viewed as job requiring time and resources for its completion. Until recently, planning was seldom used in the design phase but after technological development took place at a very rapid speed and designing became more complex with more inter departmental dependence & interaction so there was needed planning in the development phase.

10.2 Project Characteristics

Few of the important characteristics of a project are:

i. It is not a permanent entity, it is usually a non-repetitive task.

ii. It involves heavy investments.

iii. Its end results are pre decided and objectives very much clear.

iv. It can be of very long duration also, six months to 5 years or may be longer up to 10 years also.

v. It can be broken down into identifiable activities which need resources and certain time period for completion.

vi. There is a schedule for its completion with target date.

10.3 Project Management Phases

Project management may be divided into major three phases

a. Planning of Project

The planning phase of a project starts with splitting the project into smaller projects, these smaller projects is further divided into activities, and analysis of these activities by different departments takes place. The relationship of these activities with respect to each other is defined and corresponding authorities and responsibilities are stated. In this manner, possibility of overlooking of any activity necessary for completion is reduced.

b. Scheduling of Project

The main objective of the scheduling phase is to decide time completion chart for each activity (start & finish time) and relation between their time completion. Also critical activities are decided and these activities should be given proper attention so that the project can be completed within stipulated time period. For non-critical activities, slack / float times must be shown so that they can be used advantageously when such activities are delayed or limited resources are to be utilized effectively.

c. Resources Allocation

To achieve the desired & predefined objectives, resources should be allocated effectively. Resources like manpower, machines, equipments, finance, space etc. are limited in amount, so these should be effecting the limitation on time of project.

When resources are limited, they may be demanded simultaneously by different users, therefore systematic allocation of resources becomes urgent for effective use of resources to complete the project. Choice of resource usually incurs a compromise and the choice of this compromise depends on the judgment of managers.

d. Controlling of Project

It is the final phase of project management. By applying scientific methods of management in the application areas, critical operations are controlled / completed within time completion targets. By identifying key result areas, their progress through various activity progress charts and activity completion charts, a better financial as well as technical control of the project is done.

10.4 CPM & PERT Development

In the years 1956~1958, CPM & PERT techniques were developed almost simultaneously. CPM technique was developed by Walker initially to solve project scheduling company problems and it was extended to more advanced status by Mauchly Associates. In simultaneous time period, PERT was developed by engineers of US Navy working on Missile programme. It was a major project involving a lot of departments with lot of interrelated activities with unknown time frames but project was to be completed in the stipulated time period. To coordinate the activities of various departments, the group used PERT.

Both the techniques are very much similar, network oriented, using the same principle. These are time oriented method to determine the time schedule of the project. The significant difference between the two approaches is that the time estimates in CPM are assumed to be more deterministic while in case of PERT, these are described probabilistically. Now, these days CPM & PERT comprise actually one technique and difference, if any are only historical. Therefore, these techniques are referred to as 'project scheduling techniques.

These techniques may be applied in a lot of areas

- Construction of a dam, oil refinery, coal mining etc.
- Maintenance of industries, electrification etc.
- Designs of supersonic planes, space ships etc.
- Cost control.
- Designing a prototype of machine.

10.4.1 Few Definitions

It is important to understand few definitions for network presentation. These are

- **Activity** An operation which utilizes resources and consumes time has an end and a beginning is called an activity. An arrow is used to represent an activity in which its head shows the direction of progress of project. These are classified into four following categories

  a. **Predecessor activity:** Activities that must be completed immediately prior to start of another activity.

  b. **Successor activity:** Activities that cannot be started until one or more other activities are completed but immediately succeed them are called successor activities.

  c. **Concurrent activity:** Activities that can be accomplished concurrently are concurrent activities. It may be noted that the activity can be a predecessor or a successor to an event or it may be concurrent with one or more of other activities.

  d. **Dummy activity:** an activity that does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.

  Dummy activity is inserted in the network in following two situations

  a. To make activities with common starting and finishing points distinguishable.

  b. To identify and maintain the proper precedence relationship between activities which are not connected by events.

10.4.2 Rules for drawing Network Diagram

Following are the rules for drawing network diagram.

Rule no. 1 One arrow is used to represent one activity only in the network. One activity can be shown twice in the network.

Rule no. 2 No two activities can be identified by same end events.

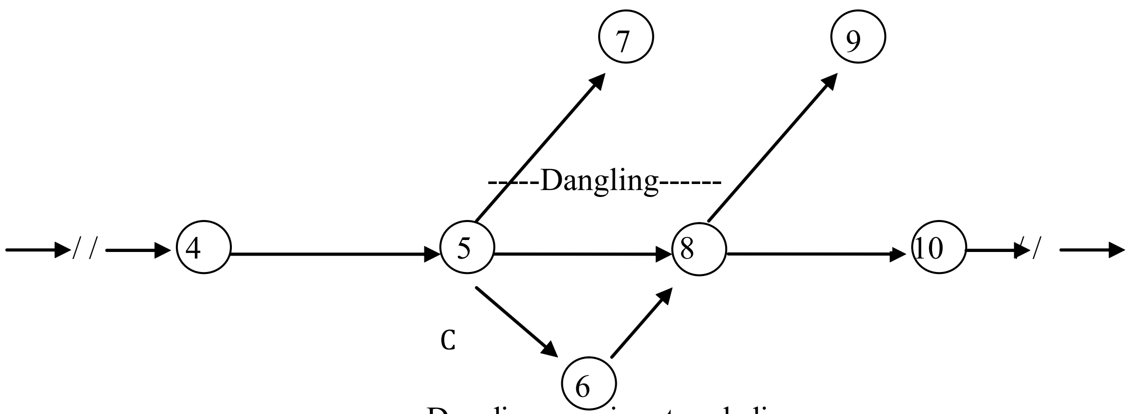
Rule no. 3 For any activity A to add in a network, following questions must be asked to ensure right precedence relationship in the network

- What activity can be finished before activity A is being started?
- Which activity will follow activity X?
- What are the activities which can occur simultaneously with this activity?

10.4.3 Errors

There are mainly three types of errors as explained below

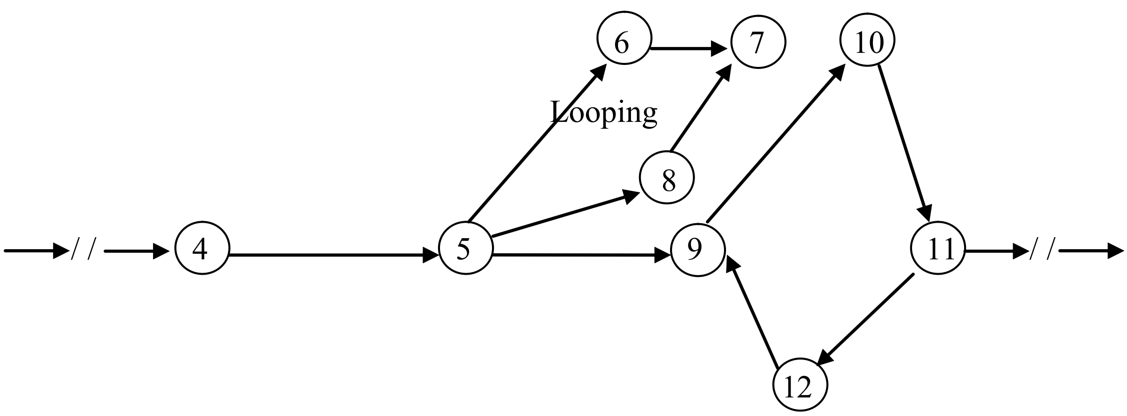
Dangling When there is disconnection of any activity before all other activities are completed is called dangling.



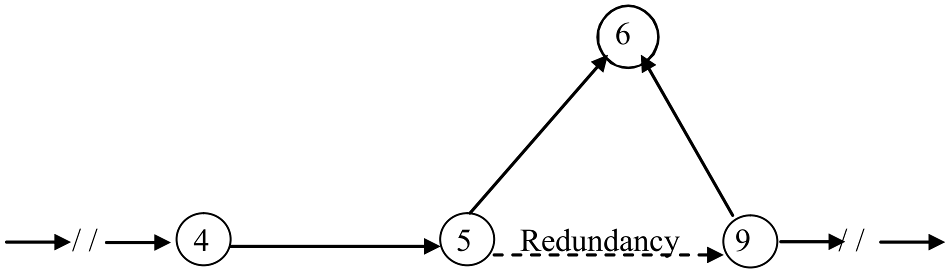
Dangling error in network diagram

For e.g. activities (5-7) and (8-9) are not the last activities in the above network diagram. So the diagram is wrong and shows the error of dangling.

Looping (or cycling) Looping error is also known as cycling error in a network diagram. When cycle in a network diagram is endless, the error is called an error of looping.



Redundancy If dummy activity is inserted unnecessarily in a network diagram, is known as the error of redundancy.



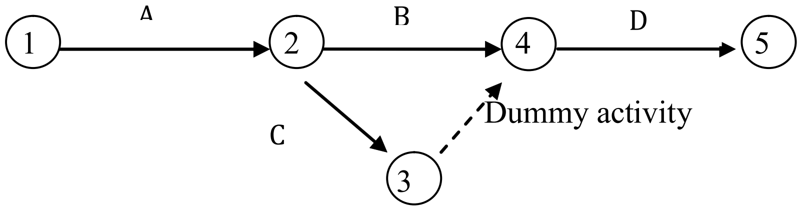
Redundancy Error diagram

10.4.4 Presentation

While doing scheduling of any project with the help of CPM / PERT techniques, the first step is to make a diagrammatical layout of activities with precedence relationships among activities.

To understand above terms, we can take an example

Considering A & B concurrent activities, C is dependent on A and D is dependent on A & B both. Such a situation can be handled by Dummy activity.

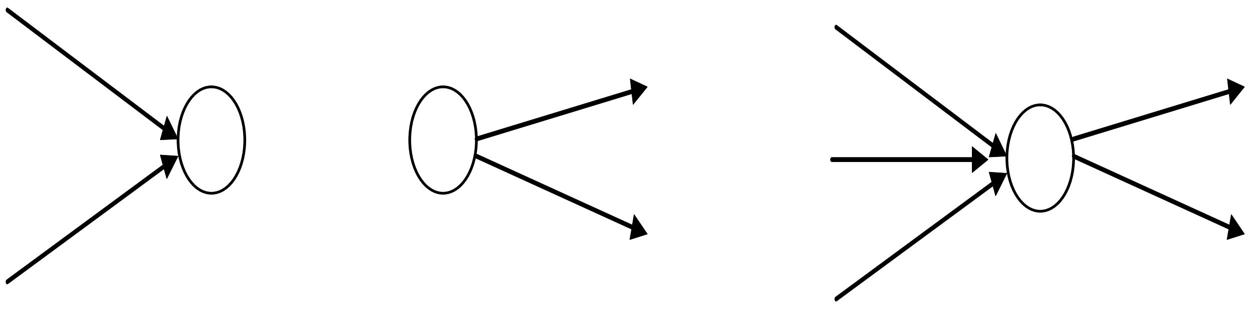


- **Event** An event may be regarded as a point in time which signifies the completion of some activities and beginning of the new ones. This is usually represented by a circle 'O' in a network and can be named as node or connector also. There are three types of events defined

  a. **Merge Event:** When there is more than one activity which is joining an event.

  b. **Burst Event:** When there are two or more than two activities leaving an event.

  c. **Merge & Burst event:** An activity may be a merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to other set activities, it may be a burst event.



Merge Event Burst Event Merge & Burst Event

An activity is the actual performance of a task which requires time and resources for its completion, it is represented by an arrow and time period of activity is represented on the arrow whereas an event is that particular instant of time at which some specific part of a project has been or is to be achieved, it is represented by a circle and numerical in the circle to show particular instance.

Examples of an event Document made, Design completed, Nail fixed

Examples of an activity Assembly of a product, preparing schedule, mixing of concrete, sand etc.

Start event   End event

- **Sequencing** A network is developed by adjoining activities, events and their sequencings. The first step in the development is to make a precedence relationships, Following points can be referred

a. To find out job to precede

b. To find out concurrent running job / jobs

c. To find out jobs to follow

d. To find out controls in the start and at the end

Few dependency relationships are given below for the construction of a network diagram, the alphabets stand for activities

i. A and B control F; C and B control G;

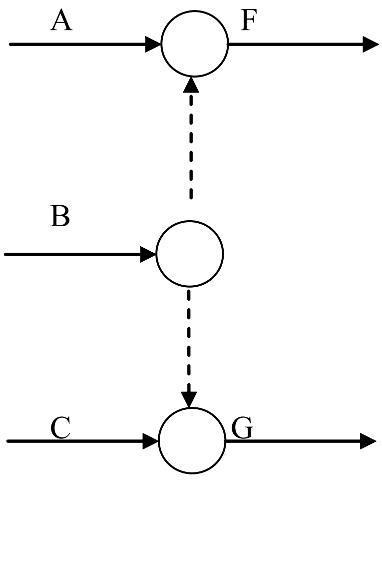
ii. A and B control F; B controls G while C controls G & H.

iii. A controls F & G; B controls G while C controls G & H.

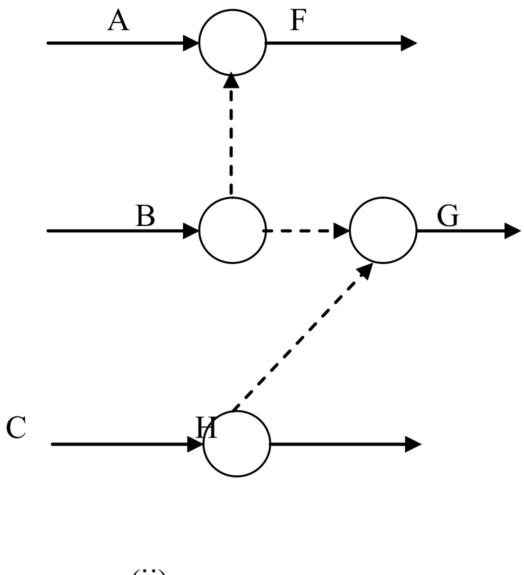
iv. F & G are controlled by A; G & H are controlled by B with H controlled by B & C.

v. A controls F,G & H; B controls G & H with H controlled by C

Sol. The above dependency relationships can be depicted by figures as given below, the dummy activities are used to find the right solution



(i)



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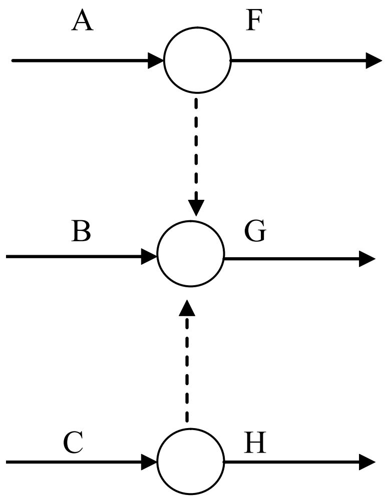
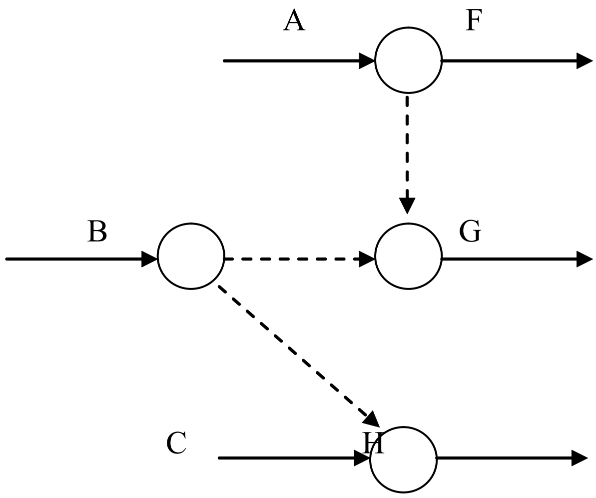
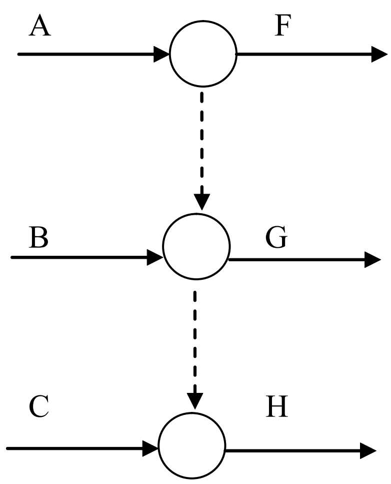


(iii)



(iv)



... (v)

Example 1. New type of water motor is to be designed for a company. Major specifications are provided and control of project is required, we can divide the project into major activities

- Drawings preparation and approval
- Cost analysis
- Tooling feasibility report
- Tools manufacturing
- Favorable cost
- Raw materials procurement
- Sub assembly order

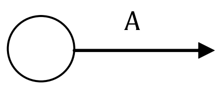
- Receiving of sub assemblies
- Manufacturing of parts
- Final assembly
- Testing & shipment

We can make an activity precedence relationship table by using the above information

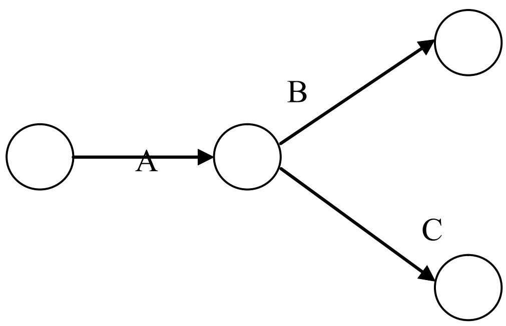
Activity	Description	Preceding Activity
A	Drawings preparation and approval	--
B	Cost analysis	A
C	Tooling feasibility report	A
D	Tools manufacturing	C
E	Favorable cost	B,C
F	Raw materials procurement	D,E
G	Sub assembly order	E
H	Receiving of sub assemblies	G
I	Manufacturing of parts	D,F
J	Final assembly	H,I
K	Testing & shipment	J

The above data can be used for making network diagram.

Step I Activity A has no preceding activity and is represented by arrowed line.

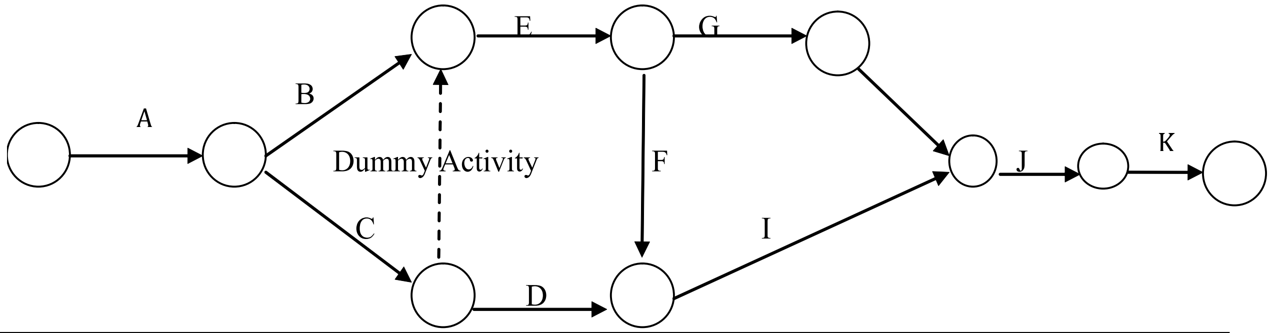


Step II Activities B & C are done concurrently and are preceded by activity 'A'.



Step III : Activity D can be sequenced after C and activity E to be scheduled after B & C. Also activity "Cost favorable cost" doesn't consume any resource and time, therefore can be considered as Dummy activity which is shown by dotted line Remaining steps discussed initially.

...

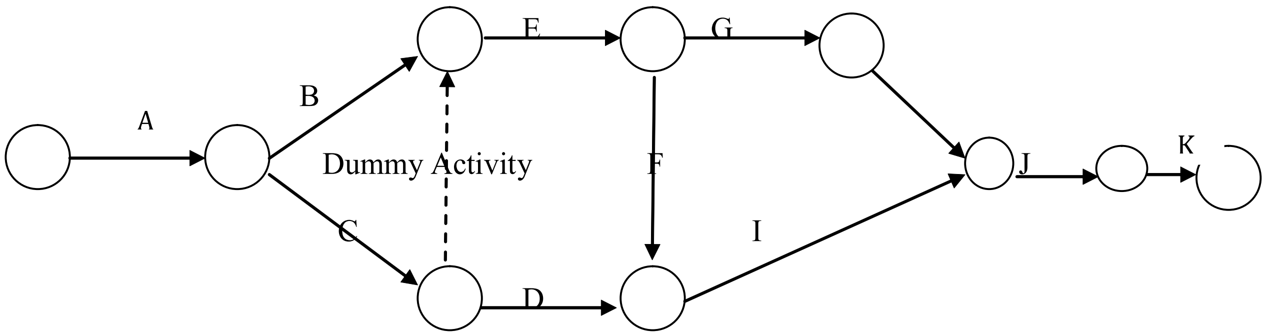


10.5 Labelling by Fulkerson's 'I-J' Rule

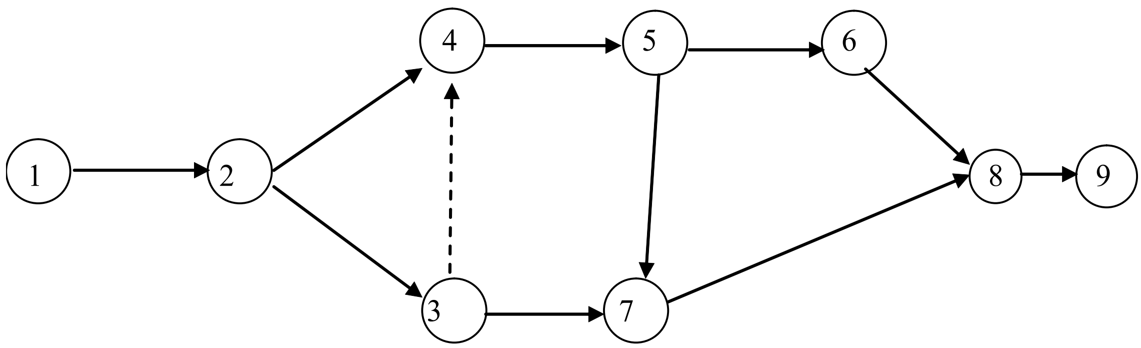
Networks are given labeling in a proper manner for convenience, to understand it very fast, to show systematic presentation. A procedure was developed for that which is known as 'I-J rule' developed by D.R. Fulkerson. Main steps of the procedure are

1. Network starts with a starting event which has arrows emerging out of it and none of the arrow entering into it. Starting event can be given no.1.
2. Delete all arrows from all numbered events. This will create at least one new start event out of preceding events.
3. Number all new start events '2', '3' and so on but numbering from top to bottom may facilitate other users in reading the network when there are more than one new start events.
4. Steps 2 & 3 are repeated till the end of network reaches.

The network shown below...



..... can be applied Fulkerson's '1-J' rule to give nodes numbering



By using Fulkerson's '1-J' rule in above network, numbering of nodes1 & 2 is palpable. By applying step 2, bottom node is the only node through which job is emerging but none entering from the top one. This is node 3. Applying step 2 again, node number 4 & 5 are easily obtained. Using step 2 again, either of the node could be numbered 6, as there are two starting points. 4, 5 & 6 being in the same row, top node can be numbered as 6.

Remaining network is simple and can be made by applying previous discussed rules.

10.6 Critical Path Analysis

After constructing network, time planning becomes essential for planning different activities and hence the project. An activity-time analysis for different activities is the time duration which activities take for completion. In network, length of an arrow is not directly proportional to the time taken by it.

This is called as schedule planning and should include following factors

i. Total completion time of project.

ii. Earliest time in starting an activity

iii. Latest time to start the activity without delay in duration of completion of project.

iv. Float of every activity (the time duration for which an activity can be delayed without delaying the completion of project)

v. Identifying critical activities and critical path.

10.6.1 Basic notations for computations

T_E / E_i Earliest occurrence time of event i

T_L / L_j Latest allowable occurrence time of event j

D_{ij} Estimated completion time of activity (i,j)

$(E_s)_{ij}$ Earliest starting time of activity (i,j)

$(E_f)_{ij}$ Earliest finishing time of activity (i,j)

$(L_s)_{ij}$ Latest starting time of activity (i,j)

$(L_f)_{ij}$ Latest finishing time of activity (i,j)

(i,j) Activity (i,j) with tail event i and head event j

Forward Pass computations / calculations (for Earliest event time) We can find out earliest start and earliest finish time for each activity (i,j) by these computations.

Step 1 It starts with 'start' node and moves to 'end node' with assumption of earliest occurrence time of zero for earliest event.

Step 2

- Earliest start time of activity (i,j) is the earliest start time of previous activity + time of that activity $(E_s)_{ij} = E_i + D_{ij}$
- Earliest finish time of activity (i,j) is the earliest start time + the activity time

  $(E_f)_{ij} = (E_s)_{ij} + D_{ij}$ or $(E_f)_{ij} = E_i + D_{ij}$

- Earliest event for event j is the maximum of the earliest finish time of all activities into that event

  $(E_f)_{ij} = (E_s)_{ij} + D_{ij}$ or $(E_f)_{ij} = E_i + D_{ij}$

Backward Pass computations / calculations (for Latest Allowable time) The latest times (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of project.

Step 1 We assume E = L for ending event.

Step 2 Latest finish time for activity (i,j) is equal to the latest event time of event j, i.e. $(L_f)_{ij} = L_j$

Step 3 Latest starting time for activity (i,j) = the latest completion time of (i,j) – activity time

$$\text{or} \quad (L_s)_{ij} = (L_f)_{ij} - D_{ij}$$

$$\text{or} \quad (L_s)_{ij} = L_j - D_{ij}$$

- **Step 4** Latest event time for event i is the minimum of the latest start time of all activities originating from that event i.e.

  $L_i = \min. [(L_s)_{ij}$ for all immediate successors of (i,j)]

  $= \min.[(L_f)_{ij} - D_{ij}] = \min.[L_j - D_{ij}]$

10.7 Floats and Slack Times

Total Float: The time duration by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

$(T_f)_{ij}$ = Latest start – Earliest start for activity (i - j)

$(T_f)_{ij} = (L_s)_{ij} - (E_s)_{ij}$

Free Float: The time duration by which the completion of an activity can be delayed beyond the earliest finish time without affecting the float of succeeding activities.

$(F_f)_{ij} = E_j - E_i - D_{ij}$

Independent Float: The time duration by which the start of an activity can be delayed without affecting the earliest start time of following activities assuming that the preceding activities have finished at its latest finish time.

$(I_f)_{ij} = E_j - E_i - D_{ij}$

Event Slacks: For any given event, the event slack is defined as the difference between the latest event and earliest event times. Mathematically for a given activity (i,j),

Head event slack = $L_j - E_j$

Tail event slack = $L_i - E_i$

All the floats can be represented in terms of head and tail event slack also

Total float = $(I_f)_{ij} = L_j - E_i - D_{ij}$

Free float = $E_j - E_i - D_{ij} = (L_j - E_i - D_{ij}) - (L_j - E_j)$

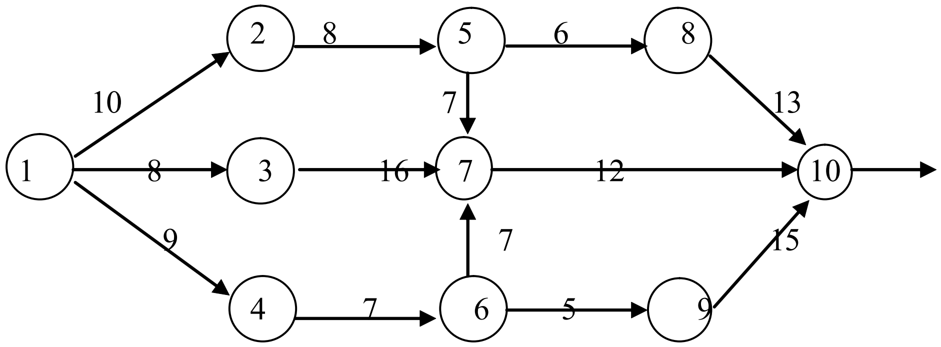
= Total float – Head event slack

Independent float = $E_j - L_i - D_{ij} = E_j - E_i - D_{ij} - (L_i - E_i)$

$= \text{Free float} - \text{Tail event slack}$

Example2. Find out Early start time (T_E) and late start (T_L) in respect of all node points and draw critical path in respect of the following network.

Sol.



Earliest Start & latest finish time for different tasks

$E_1 = 0, \ L_1 = 0$

$E_2 = E_1 + D_{12} = 0 + 10 = 10,$

$E_3 = E_1 + D_{13} = 0 + 8 = 8,$

$E_4 = E_1 + D_{14} = 0 + 9 = 9,$

$E_5 = E_2 + D_{25} = 10 + 8 = 18$

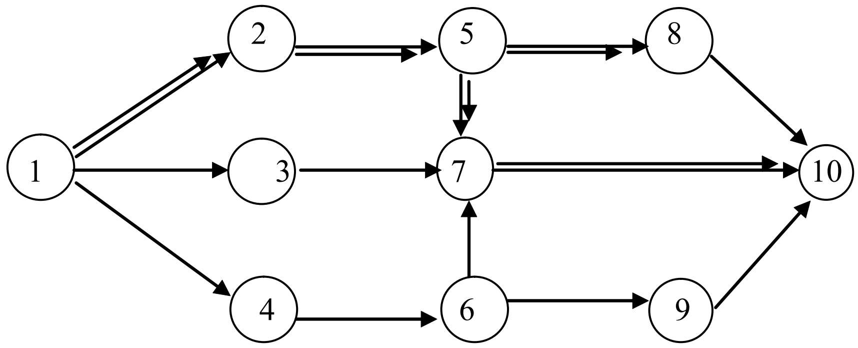
$E_6 = E_4 + D_{46} = 9 + 6 = 15$

$E_7 = \max.[E_5 + D_{57}, E_3 + D_{37}, E_4 + D_{46} = \max.[18 + 7, 8 + 16, 9 + 7] = 25$

$E_8 = E_5 + D_{58} = 18 + 6 = 24$

$E_9 = E_6 + D_{69} = 16 + 5 = 21$

$E_{10} = \max.[E_7 + D_{7,10}, E_8 + D_{8,10}, E_9 + D_{9,10} = \max.[25 + 12, 24 + 13, 21 + 15] = 37$



Calculation of L_i (Backward)

$L_{10} = E_{10} = 37$

$L_9 = L_{10} - D_{9,10} = 37 - 15 = 22$

$L_8 = L_{10} - D_{8,10} = 37 - 13 = 24$

$L_7 = L_{10} - D_{7,10} = 37 - 12 = 25$

$L_6 = \min.[L_j - D_{ij}]$, where $j = 7,9$

$= \min.[25 - 7, 21 - 5] = 1$

$L_5 = \min.[L_j - D_{ij}]$, where $j = 7,8$

$= \min.[25 - 7, 24 - 6] = 18$

$L_4 = 25 - 8 = 17$

$L_3 = 25 - 16 = 9$

$L_2 = 18 - 8 = 10$

$L_1 = 25 - 8 = 17$

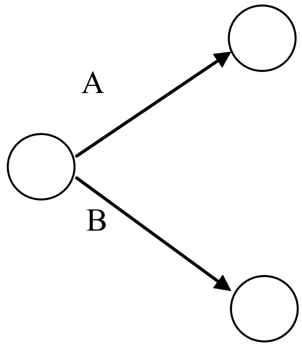
Activity (i,j)	Normal time ($D_{i,j}$)	Earliest time		Latest time		Float time ($L_j - D_{ij}$) - E_i
		Start (E_i)	Finish ($E_i + D_{i,j}$)	Start ($L_j - D_{ij}$)	Finish (L_j)	
(1,2)	10	0	10	0	10	0
(1,3)	8	0	8	1	9	1
(1,4)	9	0	9	1	10	1
(2,5)	8	10	18	10	18	0
(4,6)	7	9	16	10	17	1
(3,7)	16	8	24	9	25	1
(5,7)	7	18	25	18	25	0
(6,7)	7	16	23	18	25	2
(5,8)	6	18	24	18	24	0
(6,9)	5	16	21	17	22	1
(7,10)	12	25	37	25	37	0
(8,10)	13	24	37	24	37	0
(9,10)	15	21	36	22	37	1

Example 3. Draw a network for the simple project of erection of steel works for a shed. The various activities of the project are as under

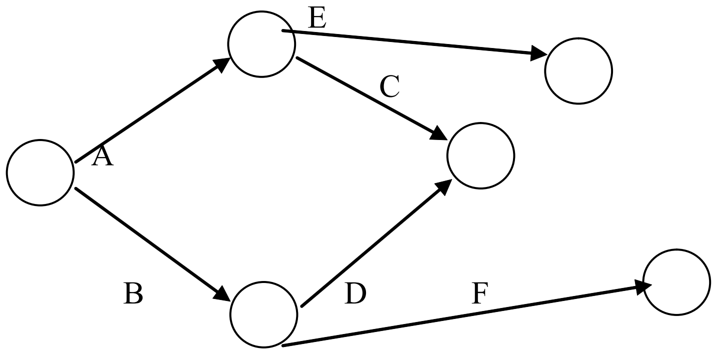
Activity	Description	Preceded by
A	Erection of site workshop	---
B	Fencing site	---
C	Bend reinforcement	A
D	Digging foundation	B
E	Fabrication of steel work	A
F	Installation of concrete pillars	B
G	Placing reinforcement	C,D
H	Concrete foundation	G,F
I	Erection of steel work	E
J	Painting of steel work	H,I
K	Giving finishing touch	J

Sol.

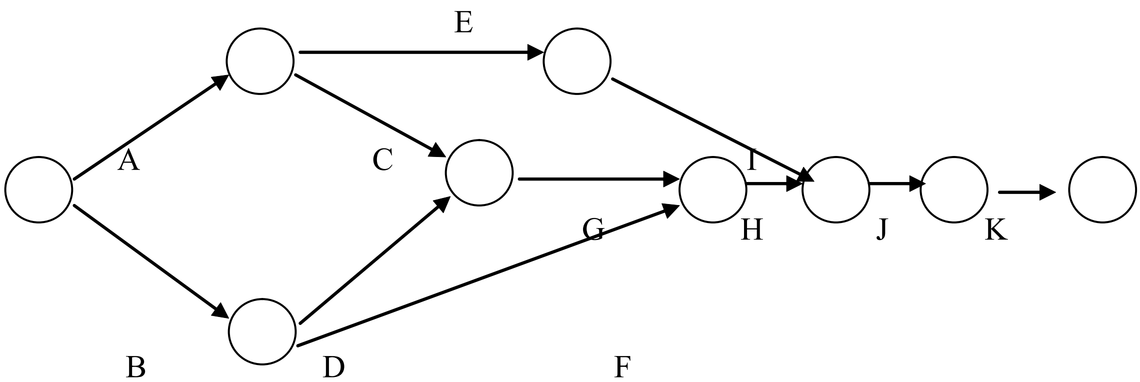
i. Activities A & B are without preceding activities



ii. Activities C & E start after A has finished, activities D & F start after B has finished



iii. G,F are preceding to H and have common event. Similarly H, I activities are preceding to J and have common event.



Example 4. Draw a network for the following activities of a project and number the events according to Fulkerson's rule

A is start event and K is the end event

A precedes event B,

J is successor event to F,

C & D are successor event to B,

D is the preceding event to G,

E & F occur after event C,

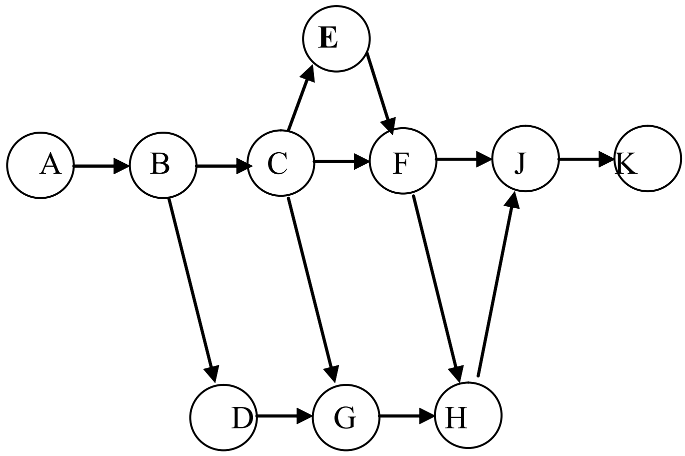
E precedes F,

C restraints the occurrence of G and G precedes H,

H precedes J & K succeeds J,

F restraints the occurrence of H.

Sol. The events A, B, C, D, E, F, G, H, J & K constitute the network



Activity G depends both on C & F, also L depend upon C hence dummy activity D_1 is used. Activity I depends both on H & L, therefore dummy activities D_2 & D_3 are used Activity N depends both on H & M hence dummy activity D_4 is used.

10.7.1 Limitations

Following are the limitations of CPM

- It does not incorporate statistical analysis in determining time estimates.
- It operates on the assumption that there is a precise known time that each activity in the project will take but this may not be a true assumption in real life.
- It is difficult to use CPM as controlling device for the reason that one must repeat the entire evaluation of the project each time changes are introduced into the network.

10.8 PERT

PERT approach takes uncertainty into account which is associated with the completion of job. In this method / approach, there are three time values are associate

- Optimistic value
- Pessimistic value
- Most likely value

Optimistic time is the shortest possible time, the activity takes for completion. It assumes that everything goes very well. It is denoted by t_o.

Most likely time is the estimation of the time that is taken normally by the activity. It assumes normal delays. It is denoted by t_m.

Pessimistic time is the maximum time taken by the activity when everything goes (assumes to be) wrong. It is denoted by t_p.

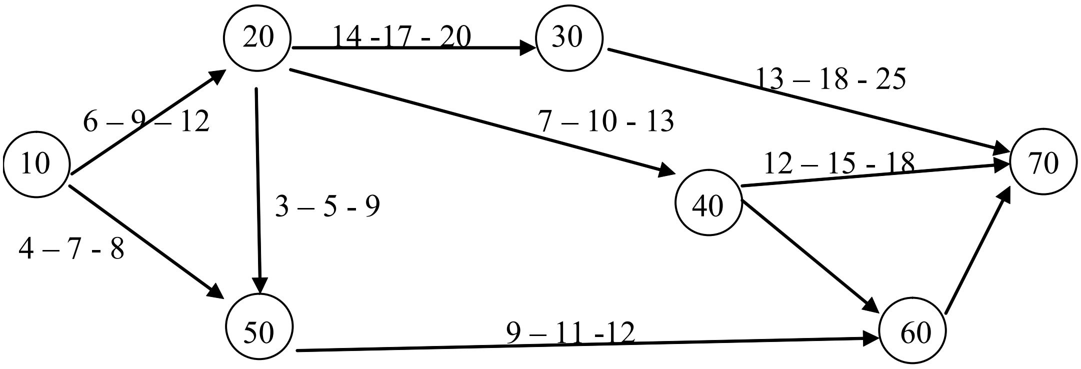
Expected time is the average time an activity will take if the activity were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution. It can be represented by the formula mentioned below

$$t_e = (t_o + 4t_m + t_p) / 6$$

Variance (σ^2) for an activity can be explained by the formula

$$\sigma^2 = [(t_p - t_o) / 6]^2$$

Example 7. Network diagram for activities is given in which three time estimates t_o, t_m, t_p are given in the order of $(t_o - t_m - t_p)$ Find out variance and expected time for each activity.



Sol. Events can be tabulated in the table

Activity i-j		t_o	t_m	t_p	$\sigma = [t_p - t_o / 6]$	$t_e = (t_o + 4t_m + t_p) / 6$
Predecessor Event i	**Successor event j**					
10	20	6	9	12	1.00	9.0
20	50	4	7	8	0.44	6.7
20	30	14	17	20	1.00	17.0
20	40	7	10	13	1.00	10.0
20	50	3	5	9	1.00	5.33
30	70	13	18	25	4.00	18.33
40	60	10	14	16	1.00	13.67
40	70	12	15	18	1.00	15.00
50	60	9	11	12	0.25	10.83
60	70	17	20	25	1.78	20.33

Example 8. The time estimates (in weeks) for the activities of a PERT network are given below

Activity	t_o	t_m	t_p
1-2	1	1	7
1-3	1	4	7
1-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

a) Draw the project network and identify all the paths through it.

b) Determine the expected project length

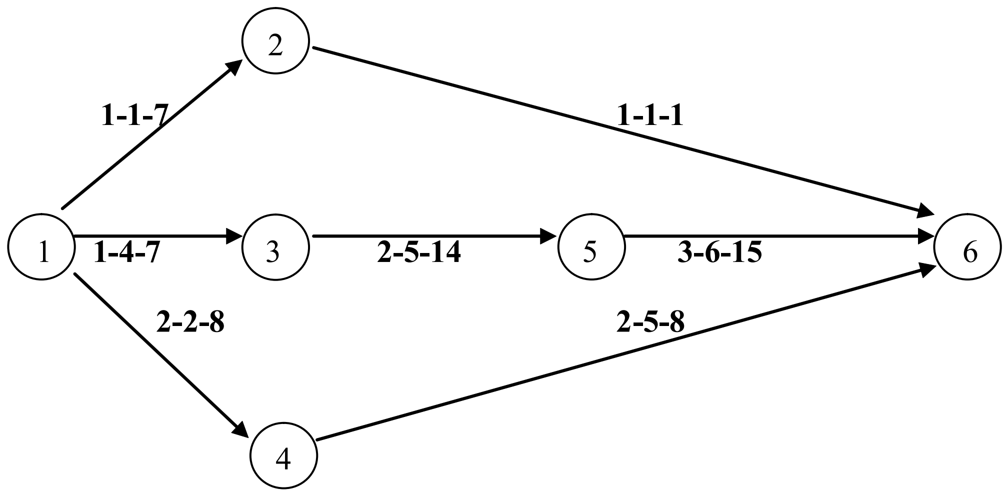
c) Calculate the standard deviation and variance of the project length

Sol. The network for the above mentioned data can be drawn as below

Various paths through the network are 1-2-5-6,

1-3-5-6,

And 1-4-6.



b) For determining the expected project length, the expected activity times need to be calculated.

Activity i-j		t_o	t_m	t_p	$\sigma = [t_p - t_o / 6]$	$t_e = (t_o + 4t_m + t_p) / 6$
Predecessor Event i	**Successor Event j**					
1	2	1	1	7	1	2
1	3	1	4	7	1	4
1	4	2	2	8	1	3
2	5	1	1	1	0	1
3	5	2	5	14	4	6
4	6	2	5	8	1	5
5	6	3	6	15	4	7

t_e = expected completion time of each activity

T_E = expected completion time of the Project.

Length of path 1-2-5-6 $2 + 1 + 7 = 10$

Length of path 1-3-5-6 $4 + 6 + 7 = 17$

Length of path 1-4-6 $3 + 5 = 8$

Since 1-3-5-6 has the longest duration, it is the critical path of network.

Therefore, expected project length $\mathbf{T_E} = 17$ weeks.

c) Variance of the project length is the sum of the variances of the activities on the critical path.

$$V_{cp} = V_{1\text{-}3} + V_{3\text{-}5} + V_{5\text{-}6} = 1+4+4 = 9$$

$$S.D = \sigma_{cp} = \sqrt{Variance}$$

$$= \sqrt{9}$$

$$= 3$$

10.8.1 Advantages and Disadvantages

Following are the advantages and disadvantages of PERT .

Advantages

- It concentrates attention on critical elements that may need correction.
- It forces managers to plan.
- It forces planning all down the line.
- It makes possible a kind of feed forward control. A delay will affect succeeding events unless the manager can somehow make up the time by shortening that of some actions in future.
- It makes possible the pressure for action at right spot and level in the organization structure at the right time.

Disadvantages

- It has emphasis on time only and not on costs.
- It is not practicable for routine planning of recurring events.
- It is not a cure-all. It will not do the planning although it forces planning.
- The calculations of probabilities is done on the assumption that a large number of independent activities operate on critical path and as such the distribution of total time is normal but this may not be true in real life which reduces the significance of all probability calculations that we make under PERT.
- It does not consider the resources required at various stages of the project.
- If time estimates to perform activities are unsatisfactory then the network diagram and the critical path will have little real meaning after project begins.

10.9 Summary

This unit gives you a detailed description about network analysis. Management of any project involves planning, coordination and control of a number of interrelated activities with limited resources such as machine, men, money and time. Furthermore, it becomes necessary to incorporate any change from initial plan as they occur and immediately know the affect of change. Therefore, the managers are forced to look for and depend on a dynamic planning and scheduling system which will produce the best possible initial plan and schedule. The quest for such a technique led to recent development of network analysis.

Network Analysis is a technique related to sequencing problems which are concerned with minimizing some measure of the system such as the total completion time of project, the overall cost and so on. The technique is useful for describing the elements in a complex situation for the purpose of designing, planning, coordinating, controlling and making decision. It is specially suited to projects which are not in routine and conducted only few times. Two most popular forms of this technique used in many scheduling situations are the Critical Path Method (CPM) and Programme Evaluation and Review Technique (PERT).

10.10 Key Words

- **Project:** A set of inter related activities which should be completed in a certain order to complete the entire project.
- **Fulkersons's '1-J' Rule:** in Networks the nodes are given labeling the procedure was developed by D.R. Fulkerson for that which is known as 'I-J rule'
- **Activity** An operation which utilizes resources and consumes time
- **Predecessor Activity:** Activities that must be completed immediately prior to start of another activity.
- **Successor Activity:** Activities that cannot be started until one or more other activities are completed.
- **Concurrent Activity:** Activities that can be accomplished concurrently
- **Dummy Activity** an activity that does not consume any kind of resource but merely depicts the technological dependence.
- **Earliest Start Time:** Earliest time at which an activity can begin.
- **Earliest Finish Time:** Earliest time at which an activity can be completed
- **Event** A point of time which signifies the completion of some activities and beginning of the new ones.
- **Optimistic Time:** is the shortest possible time when everything assumed to be going right, the activity takes for completion, denoted by t_o.
- **Most likely Time:** is the estimation of the time that is taken normally by the activity, denoted by t_m.
- **Pessimistic Time:** is the maximum time taken by the activity when everything assumed to go wrong, denoted by t_o.
- **Expected Time:** is the average time an activity will take if the activity were to be repeated on large number of times

- **Programme Evaluation & Review Technique (PERT) & Critical Path Method (CPM):** PERT and CPM are two techniques of OR which are used to show diagrammatical representation of various activities, their relations, dependencies etc.

10.11 Self Assessment Test

1. What is PERT & CPM? Explain with examples.
2. What is network analysis? When is it used? What is meant by the phrase 'Critical Path'?
3. Explain different types of Floats.
4. What are predecessors and successors, explain with examples.
5. How different types of 'errors' occur in network, explain with diagrams.
6. A project consists of series of tasks labeled A,B,——————,I with the following relationship(K < L,M means L and M can not start until K is completed, L,M < K means K can not start until both L and M are completed. Construct the network diagram having the following constraints:

   A < D,E
   B, D < F
   C < G
   B < H
   F, G <I

   Also find the optimistic time of completion of the project when the time(in days) of completion of each task is as follows

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

[Ans. 67days]

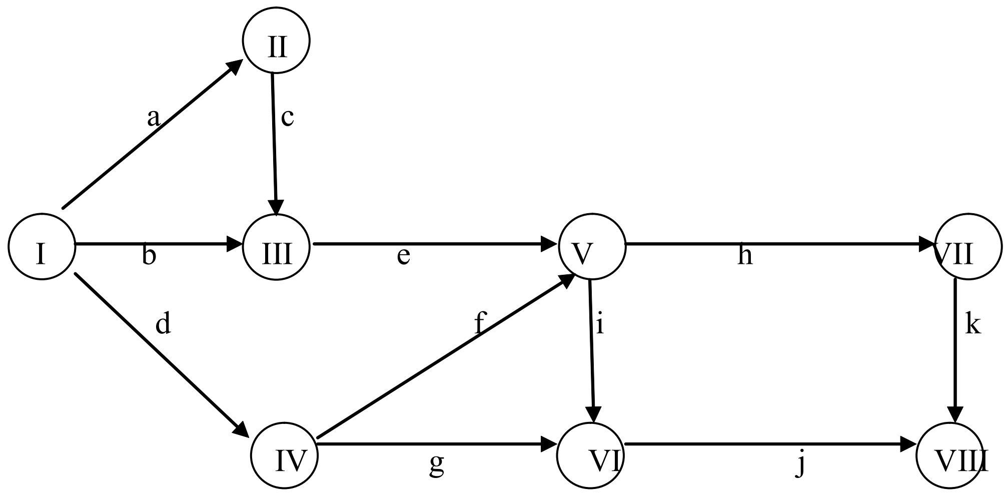
1. A project has the following time schedule

Activity	Time in weeks
(1-2)	4
(1-3)	1
(2-4)	1
(3-4)	1
(3-5)	6
(4-9)	5
(5-6)	4
(5-7)	8
(6-8)	1
(7-8)	2
(8-9)	1
(8-10)	8
(9-10)	7

Construct the network and compute

8. Find out critical path of the network given below using following data

Activity	A	b	c	d	e	f	g	h	I	j	k
Least time	4	5	8	2	4	6	8	5	3	5	6
Greatest time	8	10	12	7	10	15	16	9	7	11	13
Most likely time	5	7	11	3	7	9	12	6	5	8	9



(Ans. a-c-e-h-k)

9. A project has following details; PERT network is to be constructed. Find also Critical Path.

Activity	Most optimistic time, t_o	Most pessimistic time, t_p	Most likely time, t_m
1-2	1	5	1.5
2-3	1	3	2
2-4	1	5	3
3-5	3	5	4
4-5	2	4	3
4-6	3	7	5
5-7	4	6	5
6-7	6	8	7
7-8	2	6	4
7-9	5	8	6
8-10	1	3	2
9-10	3	7	5

(Ans. 1-2-4-6-7-9-10)

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Unit - 11 : Queuing and Game Theory

Unit Structure:

11.0 Objectives

After studying this unit, you should be able to understand

- Concept of queuing system
- Queuing Models(M/M/1, M/M/k)
- Concept of Games Theory
- Types of Games

11.1 Introduction

Queues are very common in everyday life. We quite often face the problem of long queues for a bus, banks, barber shop, a movie ticket and for various other situations. In large cities, long queues are seen in front of railway booking offices, post offices, automobiles waiting at service stations, ships waiting for berths and patients waiting for doctors. Queues are thus a common phenomenon of modern civilized life. The theory of queuing models has its origin in the work of A.K.Erlang, a Danish Engineer of the Copenhagen Telephone Company during 1910s.

Decision situations frequently arise in which units arriving for service must wait before they can be serviced. If the laws governing arrivals, service times and the order in which the arrival units are taken into service are known then the nature of the waiting situation can be studied mathematically in a simplistic way.

A queuing problem arises when a current service rate of a facility falls short of the current flow rate of customers. If a service facility is capable of servicing the customer when he arrives, no bottleneck will occur. But if it takes 10 minutes to service a customer and one customer arrives every 8 minutes, a queue will build up to infinite length if same arrival and service rate continues. In such a situation, this bottleneck is eliminated only if either arrival rate decreases or service rate increases. If size of the queue happens to be a large one then at some times people leave the queue due to impatience and a sale is lost by the concerned business unit. Hence the queuing theory is concerned with the decision making process of the business unit.

11.2 Meaning of Queue

Ordinarily the forms in front of service facilities are called a waiting line or a queue. A queue thus involves arriving customers who want to be serviced at the facility which provide the service they want to have. In short, the word queue refers to waiting in line.

The idea about a queue may be expresses as under

Customer's arrival

⇓

Waiting line/Queue

⇓

Getting serviced

⇓

Departure of customers

11.2.1Commonly Used Terminology in Queuing Theory

- **Queuing System:** A system consisting arrival of customers, waiting in queues, picked up for service, being serviced and the departure of customers.
- **Customer:** Persons arriving at a station for service. Customers may be either persons or other items.
- **Service station:** Point where service is to be provided.
- **Queuing length:** It is the number of customers waiting in the queue.
- **Waiting time** It is the time a customer spends in the queue before being serviced.
- **Number of customers in the system:** It is the sum of number of customers in the queue and number of customers being serviced.
- **Time spent by a customer in the system:** It is the sum of waiting time and service time.
- **Jockeying:** Leaving the first queue and joining the other.
- **Reneging:** Joining the queue and leaving it afterwards.
- **Balking:** Customers decides not to join the queue.

11.3 Elements of the Queuing System

A queuing system has the following elements

- Arrivals
- Service mechanism
- Queue discipline
- Output of the queue

Arrivals

Customers arrive at a service station for service. They do not come at regular intervals but an arrival into the system occurs according to some chance mechanism. Often arrival occurs at random and is independent of what has previously occurred. Arrivals may occur at a constant rate or may be in accordance with some probability distribution such as Poisson distribution, Normal distribution etc. The following information is considered relevant for input process

i) The source population

   a) Infinite (very large)

   b) Finite (limited Number)

ii) Arrival distribution

iii) Inter-arrival distribution

iv) Mean arrival rate i.e. the average number of customers arriving in one unit of time. It is represented by λ.

v) Mean time between arrivals i.e. $1/\lambda$.

Service mechanism

It is concerned with the service time and the service facilities. Service time can either be fixed or distributed in accordance with some probability distribution. Service facilities can be of following types.

Single Channel facility: In this, there is only one queue in which the customer waits till the service point is ready to take him for servicing.

One queue – several service stations facilities: In this, customer wait in a single queue until one of the service stations is ready to take them for servicing.

Several queues – one service station: In this, customer can join any one of queue but the service station is only one.

Multi channel facility: In this, there are many queues and many service stations facilities.

Multi stage channel facility: In this, customers require several types of services and there are different service stations. The customer has to get service from more than one service stations in some sequence; one after another. Each station providing a service station before leaving the system.

The following information is considered relevant for service mechanism

i) Distribution of number of customers serviced.

ii) Distribution of time taken to service customers.

iii) Average number of customers being serviced in one unit of time at a service station. It is represented by μ.

iv) Average time taken to service a customer.

Queuing discipline

Specifically it means, existence of some rule according to which the customer's actions as to when their turn comes up for the service. It may be First In First out (FIFO) or Last in Last out (LIFO). Mostly, FIFO rule is applicable in Queuing systems.

Output of the Queue

In a single channel facility, the output of the queue does not pose any problem for the customer departs after receiving the service but in multi channel facilities, output of queues becomes important.

11.4 Assumptions of the Queuing Model

- One queue
- One service station
- Queue discipline – FIFO(First In First Out)
- The population from which the queue arise is sufficiently large(assumed to be infinite)
- Arrivals and services are coming individually
- Arrivals and services occur in accordance with a Poisson process
- Time intervals between arrivals and services follow exponential distribution
- An overall assumption is that the system has been in operation for some time and has stabilized.
- Λ the average arrival rate is less than average service rate i.e. $\lambda/\mu<1$

11.5 Single-Channel Poisson Arrivals with Exponential Service Rate (M/M/1)

A single channel (1), Poisson arrivals (M) with exponential service (M) problem treats a condition where one unit is delivering the service. The inputs such as customers are considered to arrive in a Poisson manner. The exponential servicing rate is independent of the number of elements in a line. The following notations are used in queuing models

λ – mean arrival rate/unit per time

$\lambda\Delta t$ – probability that an arrival enters the system between t and $t + \Delta t$ time interval

μ - mean service rate

$\mu\Delta t$ – probability of completion of a service between t and $t + \Delta t$ time interval

n – number of customers in the system at time t

P_n – probability that exactly n customers are in the system at time t

Following formulae are used when $\frac{\lambda}{\mu} < 1$.

The expected proportion of time a facility will be idle

$P_0 = 1 - \frac{\lambda}{\mu}$ (The system is idle when there is no customer/unit in the system).

The probability that the exact number of customers n in the system at a given time

$$P(n) = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Expected number of units in the system

$$E(n) = \frac{\lambda}{\mu - \lambda}$$

Expected number of units in the queue is

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Average waiting time in the queue of an arrival is

$E(w) = E(m) / \lambda$

$$= \frac{\lambda}{\mu(\mu-\lambda)}$$

Average time an arrival spends in the system is

$$E(v) = \frac{1}{(\mu-\lambda)}$$

The probability that the number in the queue and being serviced is > k is

$$P(n>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

Average length of non-empty queue

$$E(m \mid m > 0) = \frac{\mu}{(\mu-\lambda)}$$

Average waiting time of an arrival that waits

$$E(w \mid w > 0) = \frac{1}{(\mu-\lambda)}$$

Example 1. Mean arrival rate of a queue is one customer in every 4 minutes and mean service time is 2.5 minutes. Calculate

a) average number of customers in the system
b) average queue length
c) average time a customer spends in the system
d) average time a customer waits before being served.

Sol. $\lambda = \frac{1}{4} = 0.25$ arrivals/minute

$= 15$ arrivals /hour

$\mu = \frac{1}{2.5} = 0.4$ service/minute

$= 24$ services/hour

$\frac{\lambda}{\mu} = \frac{15}{24} = 0.625$ and is less then 1.

a) Average number of customers in the system

$$E(n) = \frac{\lambda}{\mu - \lambda}$$

$$= (15/24\text{-}15) = 15/9$$

b) Average queue length

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{0.25^2}{0.4(0.4 - 0.25)} = 1.04 \text{ customers}$$

c) Average time a customer spends in the system

$$E(v) = \frac{1}{(\mu - \lambda)}$$

$$= 1/ (0.4 - 0.25) = 6.66 \text{ minutes}$$

d) Average time a customer waits before being served

$$E(w) = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{0.25}{0.4(0.4 - 0.25)}$$

$$= 4.16 \text{ minutes}$$

Example 2. In a telephone booth, arrivals are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes. Calculate

a) What is the probability that a person arriving at the booth will have to wait?
b) What is the average length of the queues that form from time to time?
c) The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least 3 minutes for the phone. By how much should the flow of arrivals be increased in order to justify a second booth?

Sol. $\lambda = 0.1$ arrivals/minute

$\mu = 0.33$ service/minute

a) Probability(an arrival has to wait) = $1 - P_0$

$$= \frac{\lambda}{\mu}$$

$$= \frac{0.1}{0.33}$$

$$= 0.3$$

a) Average length of non-empty queues

$$E(m \mid m > 0) = \frac{\mu}{(\mu - \lambda)}$$

$= 0.33/ (0.33 – 0.1)$

$= 1.43$ persons

b) Average waiting time for an arrival before gets service

$$E(w) = \frac{\lambda}{\mu(\mu - \lambda)}$$

$E(w) = 3$ Given

$3 = \lambda' / 0.33(0.33 - \lambda')$

$\lambda' = 0.16$ arrival/minute

When rate of arrival is 0.16 per minute (9.6 per hour) or more.

Example 3. Machine X break-down at an average rate of 5/hour. The breakdowns follow Poisson process. Cost of idle machine hour comes to Rs. 15/hour. Two workers A and B have been interviewed- A charges Rs. 8/hour and he services break-down machines at the rate of 7/hour whereas B charges Rs. 10/hour and he services at an average rate of 9/hour. Which worker's services should be used and why? Assume work shift is of 8 hours.

Sol. Average breakdown of machines = 5/hour

Expected number of breakdown machines in the system is

$$E(n) = \frac{\lambda}{\mu - \lambda}$$

Cost of idle machine hour = Rs 15

<u>For worker A</u>

Service rate $\mu = 7$/hour

Hourly charges = Rs. 8

$$E(n) = \frac{\lambda}{\mu - \lambda}$$

$= 5/ (7-5) = 2.5$

It means 2.5 machine hours are lost in an hour.

Total machine hours lost in a day = 2.5 X 8 = 20 machine hours.

Total cost = Hire charges of worker + Cost of idle machine

= 8 X 8 + 20 X 15

= 64 + 300 = 364

For worker B

Service rate μ = 9/hour

Hourly charges = Rs. 10

$$E(n) = \frac{\lambda}{\mu - \lambda}$$

= 5/ (9-5) = 1.25

It means 1.25 machine hours are lost in an hour.

Total machine hours lost in a day = 1.25 X 8 = 10 machine hours.

Total cost = Hire charges of worker + Cost of idle machine

= 8 X 10 + 10 X 15

= 80 + 150 = 230

Comparing the total cost for worker A and B; worker B's service would certainly be more economical.

11.5.1 Multi - Channel Poisson Arrivals with Exponential Service Rate (M/M/k)

In the multi-channel queuing system, there is k number of parallel stations; and each element in the waiting line can be served by any one station. Each service facility has the same type of service. Single waiting line is formed, and the customer moves into any one of the service stations (windows) which becomes available after having serviced a customer. The utilization factor (ρ) describes the probability of a given station being in use.

λ = Average rate of arrivals.

μ = Average service rate for each of the service stations.

k = number of service stations.

kμ= Average combined service rate of all the k service stations.

$$e = \frac{\lambda}{k\mu} = \text{utilisation factor for the system.}$$

$$\text{Also } \frac{\lambda}{k\mu} < 1$$

The probability of having no customers in a multi-channel system is given by

$$P_0 = \frac{}{\sum_{n=0}^{k-1} (1/n!)\left(\frac{\lambda}{\mu}\right)^n + \left[(1/k!)\left(\frac{\lambda}{\mu}\right)^k (k\mu/(k\mu - \lambda))\right]}$$

$$P_n = P_0\,(1/n!)\left(\frac{\lambda}{\mu}\right)^n \text{ ------------------------ when } n \le k$$

$$= P_0\left(\frac{\lambda}{\mu}\right)^n / K1.K^{n-k} \text{ ----------------------- when } n > k.$$

The probability that an arrival has to wait

$$P(n \geq k) = \frac{P_0\mu\,(\lambda/\mu)^n}{[(k-1)!(k\mu-\lambda)]}$$

Average queue length

$$E(m) = \frac{P_0\lambda\mu\,(\lambda/\mu)^n}{[(k-1)!(k\mu-\lambda)^2]}$$

Average number of customers in the system

$$E(n) = \frac{P_0\lambda\mu\,(\lambda/\mu)^n}{[(k-1)!(k\mu-\lambda)^2]} + \lambda/\mu$$

Average waiting time of an arrival

$E(w) = P_0\,\mu\,(\lambda/\mu)^k / [(k-1)!(k\mu - \lambda)^2]$

Average time an arrival spends in the system

$E(v) = P_0\,\mu\,(\lambda/\mu)^k / [(k-1)!(k\mu - \lambda)^2] + 1/\mu$

Example 4. A firm has 4 service stations to receive people with problems. Arrivals average 80 persons in an 8-hour service day to have an exponential distribution. The average service time is 20 minutes. Calculate

a) Average number of customers in the system
b) Average number of customers waiting to be serviced
c) Average time a customer spends in the system
d) Average waiting time for a customer
e) How many hours each week does a service station spend performing job?
f) Probability that a customer has to wait before he gets service.
g) Expected number of idle service station at any specified time.

Sol. $\lambda = 10/\text{hour}$

$\mu = 3/\text{hour}$

$k = 4$

$$e = \frac{\lambda}{k\mu} = 10/3\times 4 = 0.833 < 1$$

The probability of having no customers in a system is given by

$$P_0 = \frac{1}{\sum_{n=0}^{k-1} (1/n!)(\frac{\lambda}{\mu})^n + [(1/k!)(\frac{\lambda}{\mu})^k (k\mu/(k\mu-\lambda))]}$$

$$= \frac{1}{[1+\frac{\lambda}{\mu} + (1/2) * (\frac{\lambda}{\mu})^2 + (1/6) * (\frac{\lambda}{\mu})^3 + (1/24) * (\frac{\lambda}{\mu})^4 (k\mu/(k\mu - \lambda))]}$$

$$= \frac{1}{1 + 10/3 + 100/18 + 1000/162 + (10000/1944)(12/2)}$$

$$= 0.0213$$

(a) Average number of customers in the system.

$$E(n) = P_0 \lambda\mu (\frac{\lambda}{\mu})^k / [(k-1)!(k\mu - \lambda)^2] + \frac{\lambda}{\mu}$$

$$= 0.0213*10*3(10/3)^4 / [3! *(12 - 10)^2] + 10/3$$

$$= 6.61 \text{ customers}$$

a) Average number of customers waiting to be serviced

$$E(m) = P_0 \lambda\mu (\frac{\lambda}{\mu})^k / [(k-1)!(k\mu - \lambda)^2]$$

$$= 0.0213 * 10 * 3(10/3)^4 / [3! *(12 - 10)^2]$$

$$= 0.0213 * 154.1$$

$$= 3.28 \text{ customer}$$

b) Average time a customer spends in the system

$$E(v) = P_0 \mu (\frac{\lambda}{\mu})^k / [(k-1)!(k\mu - \lambda)^2] + \frac{1}{\mu}$$

$$= 0.0213 * 3(10/3)^4 / [3! *(12 - 10)^2] + 1/3$$

$$= 0.661 \text{ hour}$$

a) Average time a customer waits before being served (waiting in the queue)

$$E(w) = P_0 \mu (\frac{\lambda}{\mu})^k / [(k-1)!(k\mu - \lambda)^2]$$

$$= 0.0213 * 3 *(10/3)^4/[3! *(12 - 10)^2]$$

$$= 0.328 \text{ hour}$$

b) Total time spent each week by a service station

The utilization factor $\rho = \frac{\lambda}{k\mu}$

$$= 10/(4*3) = 0.833$$

Expected time spent in servicing customers during an 8-hour day

$= 8 \times 0.833 = 6.66$ hours

Therefore , on an average a service station is busy 33.3 hours based on a 40 hour week.

e) Probability that a customer has to wait

$P(n \geq k) = P_0\mu (\lambda/\mu)^n / [(k-1)!(k\mu - \lambda)]$

$= 0.0213 * 3 * (10/3)^4 / [3! * (12 - 10)^2]$

$= 0.6571$

f) Expected number of idle service stations at any specified time

$$P_n = P_0 (1/n!)(\frac{\lambda}{\mu})^n$$

$P_0 = 0.0213$

$P_1 = (1/1!)(10/3)*0.0213 = 0.0709$

$P_2 = (1/2!)(10/3)^2 *0.0213 = 0.1182$

$P_3 = (1/3!)(10/3)^3 *0.0213 = 0.1314$

$P_0 = 4$ Service Stations

$P_1 = 3$ Service Stations

$P_2 = 2$ Service Stations

$P_3 = 1$ Service Stations

$P_4 = 0$ Service Stations

Therefore, Expected number of idle service stations at any specified time= $4P_0 + 3P_1 + 2P_2 + 1P_3 + 0P_4$

$= 4(0.0213) + 3(0.0709) + 2(0.1182) + 1(0.1314) + 0$

$= 0.666$

Thus on average, less than one service station is idle at any time.

11.6 Game Theory

The term 'game' represents a conflict between two or more parties. Game theory is really the "science of conflict". Professor John von Neumann and Oscar Morgenstern published their book entitled "The Theory of Games and Economic Behaviour" in which they provided a new approach to many problems involving conflicting situations. Game theory is not concerned with finding an optimal strategy for a particular conflict situation but it provides general rules concerning the logic that underlines strategic behaviour of all types. Game theory applies to competitive situations are known as competitive games. To be termed as games, situations must possess the following properties.

a) Number of competitors is finite.

b) There is a conflict between the participants.

c) Each of the participants has several choices as to his appropriate actions.

d) The rules governing these choices are specified and known to all players.

e) The outcome of the game is affected by choices made by all the players. The choices are to be made simultaneously so that no competitor knows the opponent's choice until he is already committed to his own.

f) The outcome for all specific sets of choices by all of the players and, known in advance by every player.

11.6.1 Standard Convention in Game Theory

Game theory has the following set of conventions

a) Omit a description like 'player A wins five points' and replace it with integer 5.

b) Player A has choices between the rows and player B has choices between the columns. Each of the rows is a strategy for A to choose, similarity each of column is a strategy to choose from.

c) Games are represented in the form of a matrix. When games are expressed in this manner, the resulting matrix is known as payoff matrix.

d) The term strategy refers to the total pattern of choices employed by any player.

11.7 Types of Games

Games can be of several types given below

- **Two-person games and n-person games**

  In it players may have many possible choices for each play of the game but the number of players remains only two. In case of more than two persons, the game is known as n-person game.

- **Zero-sum and non-zero sum game**

  A zero-sum game is one in which the sum of the payments to all competitors is zero for every possible outcome of the game i.e. sum of points won equals the sum of the points lost. But in a non-zero sum game, sum of the payoffs from any play of the game may be either positive or negative but not zero.

- **Games with limited number of moves and unlimited number of moves**

In games with limited number of moves, number of moves is limited to fixed magnitude before play begins but in games with unlimited number of moves, it could be continued over an extended period of time and no limit is put on the number of moves.

- **Cooperative and non-cooperative games**

  Games in which players can negotiate are known as cooperative games and games in which players can not negotiate are known as non-cooperative games.

- **2 X 2 Two-person games and 2 X m and m X 2 games**

  Two person zero-sum games with only two choices open to each player are known as 2 X 2. Two person games in which one of the players has more than two choices of rows and columns and other player has exactly two choices is referred as m X 2 or 2 X m game respectively.

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix} \end{array} \qquad A \begin{pmatrix} 2 & 4 & -1 & 0 \\ 3 & -2 & 2 & 5 \end{pmatrix} \qquad A \begin{pmatrix} 4 & -1 \\ 3 & 4 \\ 0 & 2 \\ 2 & -2 \end{pmatrix}$$

Note: The scope of material in this unit is limited to 2 person Zero Sum Games only.

Example 5: Find out value of the game from the following payoff matrix and also determine the optimum strategies for the two players P1 and P2

		Player P2				Minimum Gain of A	Maximum of Minimum
	A1	3	-1	4	-2	-2	-2 ←
Player P1	A2	-1	-3	-7	0	-7	
	A3	4	-6	2	-9	-9	
Maximum loss of B		4	-1	4	0		
Minimum of Maximum			-1 ↑				

Sol. To determine optimum strategy, we take cautious approach, assume the worst and act accordingly. The positive values in the matrix are gain to player P1 and loss to player P2.Similarly negative values in the matrix are loss player P1 and Gain to player P2.

If player P1 plays with first row strategy then player P2 will play with second column for then P2 will win one point otherwise he is to lose 3, 4 and 2 if he plays with column 1, 3 and 4 resp. If player P1 plays with second row strategy then worst would happen to him only when player P2 plays with 3rd column because in that case P2 wins 7 points. If player P1 plays with 3rd row strategy, then worst he can expect is of losing 9 points when P2 plays with 4th column. In this problem, then player P1 should adopt first row strategy because only then his loss will be minimum. Thus player P1 can make the best of the situation by aiming at the highest of these minimal payoffs. This decision rule is known as "maximin strategy"(maximising minimum Gains)

From the point of view of player P2, it can be said that if player P2 plays with column first strategy, the maximum he can lose is 4 points if player P1 adopts strategy of row 3. If player P2 plays with column two strategy there is no question of any loss whatever may be the strategy of row one for only then his loss will be minimum. If player P2 plays with column third strategy the maximum he can lose is 4 points if P1 adopts strategy of row one. If P2 adopts strategy of column 4 he can lose at the most 2 points if P1 adopts strategy of row one. In this problem then P2 should adopt second column strategy and thereby ensure a win of 1 point which is the maximum in the given case. Thus player P2 can make the best of the situation by aiming at the lowest of these maximum payoffs (viz. 4, -1, 4, 2). Thus he should seek the minimum among the maximum payoffs. This decision rule is known as "Minimax strategy".

Player P2

$$\text{Player P1} \quad \begin{bmatrix} 3 & -1 & 4 & \boxed{-2} \\ -1 & -3 & -7 & 0 \\ 4 & -6 & 2 & -9 \end{bmatrix} \quad -2 \longleftarrow \quad \text{Choice of p1}$$

Choice of P2

The Saddle point and Pure Strategy

The saddle point in a payoff matrix is one which is the smallest value in its row and the largest value in its column. The saddle point is also known as equilibrium point in the theory of games. An element of a matrix that is simultaneously minimum of the row in which it occurs and the maximum of the column it occurs is a saddle point of the game. In a game having a saddle point; optimum strategy for player P1 is always to play the row containing a saddle point and for the player P2 to always play the column that contains a saddle point. Saddle point also gives the value of such a game. Such a choice is known as PURE STRATEGY i.e. whenever a player chooses only one strategy for all repetitions of the game; and his gain will be less, if he shifts to another strategy.

Saddle point in a payoff matrix concerning a game may be there and may not be there, If there is a saddle point we can easily find out the optimum strategies and the value of the game .Its saddle point does not exist we have to use algebraic methods for working out the solutions concerning the game.

Example 6: Find the optimum strategies and the value of the game from the following payoff matrix concerning two-person game:

Player P2

$$\text{Player P1} \quad \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$$

Solution: In the given game, there is no saddle point; therefore the players will resort to what is known as MIXED STRATEGY i.e. player P1 will play each of his rows a certain portion of time and player will play each of his columns a certain part of the time. The question then is to determine what proportion of the time a player should spend on his respective rows and columns. This can be done by the use of algebraic method stated as follows:

Let Q equal the proportion of time player P1 spends playing the first row, then (1-Q) must equal the time he spends playing his second row (because one equals the time available for play). Similarly, suppose player P2 spends time P in playing first column and (1-P) proportion of time he spends playing the second column. All this can be stated as under. Player P2

$$\text{Player P1} \quad \begin{array}{c} \\ Q \\ 1-Q \end{array} \begin{array}{c} \begin{array}{cc} P & 1-P \end{array} \\ \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix} \end{array}$$

Now, we must find out the values of Q & P. Let us analyse the situation from P1's point of view. He would like to devise a strategy that will maximise his winning (or minimise his losses) irrespective of what his opponent Y does. For this P1 would like to divide his play between his rows in such a manner that his expected winnings or losses when P2 plays the first column will equal his expected winnings or losses when P2 plays the second column. (Expected winnings indicated the sum, over time, of the payoffs that will obtain multiplied by the probabilities that these payoffs will obtain). In our case, we can calculate the same as under:

P1's EXPECTED WINNINGS

	When P2 plays Column one	When P2 plays Column two
P1 plays row one Q of the time	(1) (Q)	(4) (Q)
P1 plays row two 1-Q of the time	(5) (1-Q)	(3) (1-Q)
P1's total expected winnings	Q + 5(1-Q)-----(1)	4Q + 3(1-Q)-----(ii)

Equating the expected winnings of P1 when P2 plays column one when P2 plays column two; we can find the values of Q as follows:

$$Q + 5(1\text{-}Q) = 4Q + 3(1\text{-}Q)$$
$$Q + 5 - 5Q = 4Q + 3 - 3Q$$
$$-5Q = -2$$
$$Q = 2 / 5$$
$$1 - Q = 3 / 5$$

This means that player P1 should play his first row 2/5 of the time and his second row 3/5 of the time if he wants to maximise his expected winnings from the game.

On the similar basis expected losses of P2 can be worked out as under:

P2's EXPECTED WINNINGS

	When P2 plays Column one P of the time	When P2 plays Column two 1-P of the time	P2's total expected losses
P1 plays row one	(1) (P)	(4) (1-P)	(P)+ 4(1-P)
P1 plays row two	(5) (P)	(3) (1-P)	5R + (3) (1-P)

$$P + 4(1\text{-}P) = 5P + 3(1\text{-}P)$$
$$P + 4 - 4P = 5P + 3 - 3P$$

$$-5P = -1$$

$$P = 1/5$$

$$1 - P = 4/5$$

Value of the game = Q+5(1-Q)———————(i)

Or

4Q + 3(1-Q)—————(ii)

= 2/5 + 5*3/5

= 3 2/5 Rupees.

Check (ii) also gives the same values.

This means that player P2 should play his first column 1/5 of the time and his second column 4/5 of the time if he wants to minimise his expected losses in the game.

Now, we can illustrate the original game with the appropriate strategies for each of the player as follows:

Player P2

$$\begin{array}{ccc} & & \begin{array}{cc} 1/5 & 4/5 \end{array} \\ \text{Player P1} & \begin{array}{c} 2/5 \\ 3/5 \end{array} & \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix} \end{array}$$

Algebric Method

Let a 2*2 game be denoted as follows.

$$\begin{array}{ccc} & & \begin{array}{cc} P1 & P2 \end{array} \\ \text{Price} & \text{Strategy} & \begin{array}{cc} B1 & B2 \end{array} \\ \begin{array}{c} Q1 \\ Q2 \end{array} & \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

Step one, Subtract the smaller payoff in each row from the larger one and the smaller payoff in each column from the larger one.

$$\begin{array}{cccc} & P2 & & \\ P1 & \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix} & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} \text{(i.e. 4-1 = 3)} \\ \text{(i.e. 5-3 = 2)} \end{array} \\ & \begin{array}{cc} 4 & 1 \end{array} & & \end{array}$$

(i.e. 5-1 = 4)   (i.e. 4-3 = 1)

Step 2: Interchange each of these pairs of subtracted numbers found in step 1 above

$$\begin{array}{cc} & \text{P2} \\ \text{P1} & \left[\begin{array}{cc} 1 & 4 \\ 5 & 3 \end{array}\right] \begin{array}{c} 2 \\ 3 \end{array} \\ & \begin{array}{cc} 1 & 4 \end{array} \end{array}$$

Step 3: Put each of the interchanged numbers over the sum of the pair of numbers

$$\begin{array}{cc} & \text{P2} \\ \text{P1} & \left[\begin{array}{cc} 1 & 4 \\ 5 & 3 \end{array}\right] \begin{array}{c} 2/(2+3) \\ 3/(2+3) \end{array} \\ & \begin{array}{cc} 1/(1+4) & 4/(1+4) \end{array} \end{array}$$

Step 4: Simplify the fraction to obtain the proper proportions or the required strategies

$$\begin{array}{cc} & \text{P2} \\ \text{P1} & \left[\begin{array}{cc} 1 & 4 \\ 5 & 3 \end{array}\right] \begin{array}{c} 2/5 \\ 3/5 \end{array} \\ & \begin{array}{cc} 1/5 & 4/5 \end{array} \end{array}$$

Now, we determine the values of the game. Looking at the game from P1's point of view we can argue as follows:

(i) During the 1/5 of the time P2 plays column one P1 wins 1 point 2/5 of the time (when P1 plays row one) and 5 points 3/5 of the time (when P1 plays row second)

(ii) During the 4/5 of the time P2 plays column two P1 wins 4 points 2/5 of the time (when P1 plays row one) and 3 points 3/5 of the time (when P1 plays row second)

Thus total expected winnings of player P1 are the sum of the above statements which is under:

$$\frac{1}{5}\{(1)(\frac{2}{5}) + (5)(\frac{3}{5})\} + \frac{4}{5}\{(4)(\frac{2}{5})+(3)(\frac{3}{5})\}$$

$$= (\frac{1}{5})(\frac{17}{5}) + (\frac{4}{5})(\frac{17}{5})$$

$$= \frac{17}{25} + \frac{68}{25}$$

$$= \frac{85}{25}$$

$$= \frac{17}{5}$$

Thus the value of the game is 17/5 which means that player P1 can expect to win an average payoff of 17/5 points for each play of the game if he adopts the strategy we have determined as stated above. If the value of the game determined above had a negative sign, it would simply signify that P2 was the winner. The same result we can also get by looking at the game from P2's point of view doing similar calculations.

Concept of Dominance to reduce the size of Game to 2×2

Example 7 . Find the optimal strategies for A and B in the following game. Also obtain the value of the game.

		B's Strategy		
		b_1	b_2	b_3
A's Strategy	a_1	9	8	-7
	a_2	3	-6	4
	a_3	6	7	-7

The pay-offs in relation to a_1 are equal to or greater than the corresponding pay-offs from a_3. Thus, a_1 dominates a_3. Deleting a_3, we get,

		B's Strategy		
		b_1	b_2	b_3
A's Strategy	a_1	9	8	-7
	a_2	3	-6	4

Now, we observe that strategy b_1 is dominated by b_2. Thus, we delete b_1. This gives the following matrix:

		B's Strategy	
		b_1	b_2
A's Strategy	a_1	8	-7
	a_2	-6	4

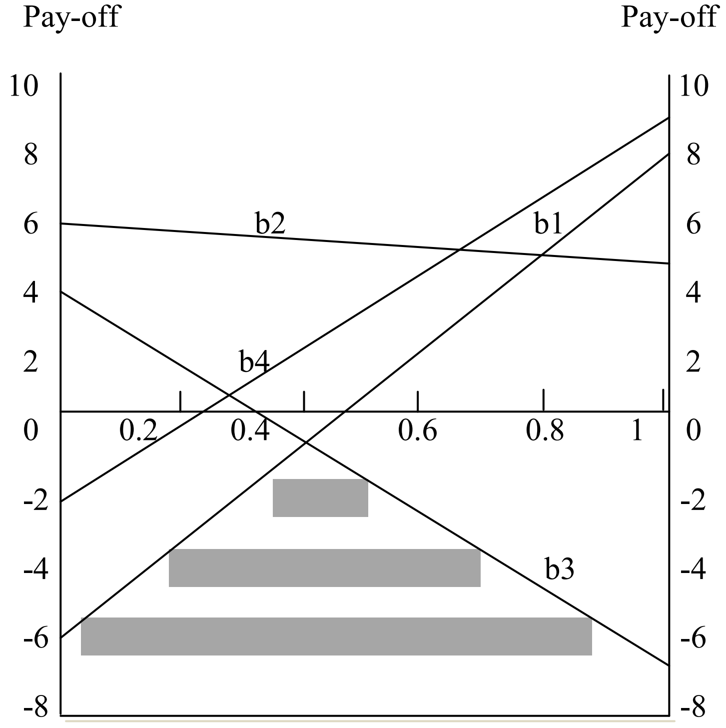
Solving 2×n or n×2 games using graphical method.

Example. Solve the following game using graphical approach:

		B's Strategy			
		b_1	b_2	b_3	b_4
A's Strategy	a_1	8	5	-7	9
	a_2	-6	6	4	-2

Here A has two strategies a_1 and a_2 which, suppose, he plays with probabilities *x* and 1-*x* respectively. When B chooses to play b_1 the expected payoff for A shall be 8x +(-6)(1-x) or 14x-6. Similarly, the expected pay-off functions in respect of b_1, b2, b_3 and b_4 can be derived as being 6-x; 4-11x; and 11x-2, respectively. We can represent these by graphically plotting each pay-offs as a function of x. This is shown in below figure.

The lines are marked b_1, b2, b_3 and b_4 and they represent the respective strategies. For each value of x, the height of the lines at that point denotes the pay-offs of each of B's strategies against (x,1-x) for A. A is concerned with his least pay-off when he plays a particular strategy, which is represented by the lowest of the four lines at that point, and wishes to choose x as to maximise this minimum pay-off. This is at K in the figure where the lower envelop (represented by the shaded region), the lowest of the lines at the point, is the .highest. This point lies at the intersection of the lines representing strategies b_1 and b_3. The distance KL = -0.4or -2/5) in the figure represents the game value, V, and x = OL(=0.4 or 2/5) is the optimal strategy for A.



Alternatively, the game can be written as a 2×2 game as follows, with strategies a1and a2 for A, and b_1, b2 and b_3 for B.

	b_1	b_3
a_1	8	-7
a_2	-6	4

This problem is the same as in above previous example

$$x = \frac{4-(-6)}{(8+4)-(-7-6)}$$

$$= \frac{2}{5}$$

$$y = \frac{4-(-7)}{(8+4)-(-7-6)}$$

$$= \frac{11}{25}$$

$$V = \frac{8X4-(-7)X(-6)}{(8+4)-(-7-6)}$$

$$= -\frac{2}{5}$$

Thus, optimal strategy for A is ($\frac{2}{5}, \frac{3}{5}$) and for B is ($\frac{11}{5}$, 0 , $\frac{14}{25}$,0).

11.8 Summary

In this unit the concept of Queue System and its difference with the Queue has been discussed. The assumptions under which the models of Queue specially the single service stations (MMI/∞ FIFO) and multiple service station (MMK/∞ FIFO) have been discussed. Their applications in practice with numerical examples have also been explained.

In Game Theory, the concept of Pure and Mixed Strategies, the concept of Saddle Point and Principles of Dominance have been explained .The applications of Games under competitive situations have been discussed. The methods of solutions for 2 X 2, 2 X n, m X 2 and the L.P formulation have also been discussed with examples.

11.9 Key Words

- **Queue:** A group of customers waiting for service in a system is a queue.
- **Waiting Time:** The time spent in the queue before being serviced.
- **Balking:** Customers decides not to join the queue.
- **Jockeying:** Leaving the first queue and joining the other.
- **Reneging:** Joining the queue and leaving it afterwards.
- **Strategy :** The numbers of competitive choices that are available for a player are called strategies to that player.
- **Payoff matrix:** it is a tabular representation of payments that should be made at the end of a game.
- **Zero-sum Game:** Games with the algebraic sum of gains and losses of all the players equal to zero are called zero-sum game.
- **n-person game:** a game having n persons.
- **Saddle Point:** is a point in the domain of a function that is a stationary point but not a local extremum (maxima and minima).
- **Pure strategy:** provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he or she could face. A player's **strategy set** is the set of pure strategies available to that player.

- **Mixed Strategy:** is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player, even if their strategy set is finite.

- **Dominance:** Since all players are assumed to be rational, they make choices which result in the outcome they prefers most, given what their opponents do. In the extreme case, a player may have two strategies A and B so that, given any combination of strategies of the other players, the outcome resulting from A is better than the outcome resulting from B. Then Strategy: A is said to *dominate* strategy B. A rational player will never choose to play a Dominated strategy.

11.10 Self Assessment Test

1. Write short note on the following

    a. Queue Discipline

    b. Queuing System

    c. Service mechanism

2. Explain the meaning of a queue and state the object of queuing analysis. Briefly describe the elements of queuing system.

3. At a one-man barbershop, it takes on the average one-half hour to service a customer and the customer arrive at an average rate of one every forty five minutes. If a Poisson exponential model is assumed then find

    a. The average time a customer spends in the shop.

    b. The average time the customer spends waiting in the queue.

    c. The expected time the barber will remain idle.

    d. At any given time, expected number of the customer in the shop.

    e. At any given time, expected number of the customer in the queue.

4. Given the following payoff matrix

Union strategies	Company strategies			
	C_1	C_2	C_3	C_4
X	+20	+15	+12	+35
Y	+25	+14	+8	+10
Z	+40	+2	+19	+5
W	-5	+4	+11	0

Determine the optimum strategies for the union as well as for the company. What is the expected gain to the union of the above game?

5. For what type of business problems might game theory be helpful? Explain.

6. The matrix of a certain two-person zero-sum 3X3 game is given below

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \end{array}$$

What should be the optimal strategies for X and Y in the above game and the value of the game?

11.11 References

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