No. of Printed Pages : **2**

Sl. No.

21MAT1C2L

M.Sc. I Semester Degree Examination, April/May - 2023

MATHEMATICS

Real Analysis

Time : 3 Hours | Maximum Marks : 70

Note : *Answer **any five** questions, with Q.No. **(1)** is **compulsory**. Each question carries **14** marks.*

1. (a) Prove-or-disprove : Q is countable set. **5+5+4**

(b) Prove that a non-empty subset of **R** bounded below has infimum in **R**.

(c) State and prove Archimedean property of Real numbers.

2. (a) Compute the following for the subsets of **R** under Euclidean topology on **R** **5+5+4**

(i) interior of **N** (ii) derived set of Z

(iii) closure of (0, 1) (iv) boundary of (0, 2)

(v) exterior [2, 3]

(b) Define compact set. Prove that every compact set in **R** is bounded.

(c) Prove-or-disprove the following statements :

(i) Infinite union of closed sets is closed.

(ii) $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$.

3. (a) Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let e be a cluster point of A. If $\underset{x \to e}{\text{lt}} f > 0$, then prove that there exists a neighbourhood $V_\delta(e)$ such that $f(x) > 0$, for all $x \in A \cap V_\delta(e)$, $x \neq e$. **5+5+4**

(b) State and prove Bolzano's Intermediate value theorem.

(c) Classify discontinuities for the following functions at mentioned points :

(i) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 3 & x = 0 \end{cases}$

(ii) Signum function at $x = 0$.

4. (a) Prove or disprove the following statement : **5+5+4**

let $\{f_n\}$ be a sequence of integrable functions and $f_n \to f$ pointwise on [0, 1].

then $\int_0^1 f_n \, dt = \int_0^1 f(t) \, dt$.

(b) State and prove Cauchy's criteria for uniform convergence of series of functions.

(c) Test the uniform convergence for the following sequence and series of functions :

(i) $f_n(x) = \dfrac{1}{x + n^2}, x \in [0, 10]$

(ii) $f_n(x) = x^n, x \in [0, 1]$

(iii) $\sum_{n=1}^{\infty} \dfrac{2^n x^{2n-1}}{1 + x^{2n}}, x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

5. (a) If $\{f_n\}$ converges uniformly to f on [a, b], and each function f_n is integrable, then prove that f is integrable on [a, b] and $\lim_{n \to \infty} \int_a^b f_n(t)\, dt = \int_a^b f(t)\, dt$. **7+7**

(b) Let $\sum_{n=1}^{\infty} f_n$ be a series of differentiable functions on [a, b] and such that it converges at least at one point $x_0 \in$ [a, b]. If the series of differentials Σf_n^1 converges uniformly to G on [a, b], then prove the following :

(i) Σf_n converges uniformly on [a, b] to f, and

(ii) $f'(x) = G(x), \forall x \in [a, b]$.

6. Discuss the compactness and connectedness for the following subsets of **R**. **8+6**

(a) Under Euclidean topology on **R**.

(i) $\sin\left(\left[0, \frac{\pi}{2}\right]\right)$ (ii) $\sin([0, 6\pi)]$ (iii) **N** (iv) Q

(b) Define connectedness. Prove that continuous image of connected set is connected.

7. (a) State and prove Weierstrass-M*test. **7+7**

(b) Let $\{f_n\}$ be a sequence of functions, such that $f_n \to f$ pointwise on [a, b] and let $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$. Then prove that $f_n \to f$ uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$.

8. (a) State and prove Cantor's theorem. **5+5+4**

(b) Prove that uniform continuous function is continuous. Discuss the converse.

(c) Define Monotone function. Prove that monotone function have no discontinuity of second kind.

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