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21MAT2C6L

Sl. No.

M.Sc. II Semester Degree Examination, October - 2023

MATHEMATICS

Linear Algebra

(NEP)

Time : 3 Hours Maximum Marks : 70

Note : *Answer **any five** questions with question Number. **1** is **compulsory**.*

1. (a) Prove that a subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

(b) Prove - or - disprove the statement : In a finite dimensional vector space V every non-empty linearly independent set of vectors is a part of basis.

(c) Find the values of a, b such that {(1, 1, 0, 0), (1, 0, 0, 1), (1, 0, a, 0), (0, 1, a, b)} is linearly independent. **5+5+4**

2. (a) If W_1 and W_2 are finite dimensional subspaces of a vector space V, then prove that W_1+W_2 is finite - dimensional and

$\dim(W_1 \cap W_2) = \dim(W_1)+\dim(W_2)-\dim(W_1+W_2)$.

(b) Define linear transformation and give at least three examples.

(c) Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear mapping defined by $T(x, y, z) = (x - y, x - z)$. Then compute a basis for ker(T). **5+5+4**

3. (a) Show that the space $L(\mathbf{R}^4, \mathbf{R}^3)$, set of all linear transformations from $\mathbf{R}^4$ into $\mathbf{R}^3$ is finite - dimensional and has dimension 12.

(b) Let V be a finite - dimensional vector space over the field F, and let W be a subspace of V. Then prove the following.

$\dim(W)+\dim(W^0)=\dim(V)$

(c) Let V and W be finite dimensional vector spaces over the field F. Prove that V and W are isomorphic if and only if $\dim(V)=\dim(W)$. **5+5+4**

4. (a) Let V be a finite - dimensional vector space over the field F, and let $B = \{\alpha_1, \alpha_2 ...., \alpha_n\}$ and $B' = \{\alpha'_1, \alpha'_2, ..., \alpha'_n\}$ be ordered bases for V. Suppose T is a linear operator on V. If $P=[P_1, P_2 ....., P_n]$ is the $n \times n$ matrix with columns $P_j = [\alpha'_j]_B$, then prove the following :

$[T]_{B'} = P^{-1}[T]_B P$.

P.T.O.

(b) Write a note on four fundamental subspaces of a $m \times n$ matrix over the field **R**.

(c) Write a note on matrix of inner product in the ordered basis $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of V. **5+5+4**

5. (a) Prove that an Orthogonal set of non-zero vectors is linearly independent.

(b) Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, 1)$, $\beta_3 = (0, 3, 4)$, to obtain an orthonormal basis for $\mathbf{R}^3$ with the standard inner product.

(c) Let W be a subspace of an inner product space V and β be a vector in V. Then prove the following :

The vector α in W is a best approximation to β by vectors in W if and only if $\beta - \alpha$ is orthogonal to every vector in W. **5+5+4**

6. (a) Prove that similar matrices have the same characteristic polynomial. Discuss the converse.

(b) Let T be a linear operator on a finite dimensional space V. let $C_1, C_2, ..., C_k$ be the distinct eigen values of T and let W_i be the null space of $(T - C_i I)$. Then prove that the following are equivalent.

(i) T is diagonalizable.

(ii) The characteristic polynomial for T is

$$f(x) = (x - C_1)^{d_1} (x - C_2)^{d_2} \ldots (x - C_k)^{d_k} \text{ and } \dim (W_i) = d_i,\ i = 1, 2, \ldots k.$$

(iii) $\dim (V) = \dim(W_1) + \ldots + \dim (W_k)$ **7+7**

7. (a) Let T be a linear operator on an n - dimensional vector space V [or, A be $n \times n$ matrix]. Then P.T characteristic and minimal polynomials for T [for A] have the same roots, except multiplities.

(b) Solve the system of equations :

$$\frac{dx}{dt} = 3x - 10y$$

$$\frac{dy}{dt} = x - 4y$$

7+7

8. (a) Determine all possible Jordan Canonical form for a matrix of order 5 whose minimal polynomial is $(x-2)^2$.

(b) Check the diagonalizability of the matrix $\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$

(c) Let V be finite dimensional vector space. Then prove that V is isomorphic to V**. **5+5+4**

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