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M.Sc. II Semester Degree Examination, September/October - 2022

MATHEMATICS

21MAT2C6L : DSC 6 : Linear Algebra

Time : 3 Hours Maximum Marks : 70

Note : *Answer **any five** of the following questions with Question No.**1** is **Compulsory**, each question carries **equal** marks.*

1. (a) If B is an $p \times q$ matrix and $p<q$, then prove that the homogeneous system of linear equations $Bx=0$ has a non-trivial solution. **5+5+4**

(b) Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_k$ in V. Prove that any independent set of vectors in V is finite and contains no more than k elements.

(c) Prove or disprove union of two subspaces is a subspace.

2. (a) Let V be n-dimensional vector space over the field F. Then show that V is isomorphic to F^n. **5+5+4**

(b) Let dim $U=m$ and dim $V=n$ over the field F. Let B and B' be two ordered bases for U and V respectively. Let $T: U \to V$ be any linear transformation. If $\alpha \in U$, then prove the following.

$$[T(\alpha)]_{B'} = [T]_B^{B'} [\alpha]_B$$

(c) Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(x, y) = (2x, \frac{y}{2})$. Show that T is a linear and find its matrix with respect to ordered basis $\{(1, 0), (0, 1)\}$.

3. (a) Let $f: V \to F$ be a linear functional. If V is finite dimensional, then prove that $\text{nullity}(f) = \dim(V) - 1$. **7+7**

(b) Let V be a finite-dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha)$, $f \in V^*$. Then prove that mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

4. (a) For $\alpha = (x_1, x_2, \ldots\ldots, x_n)$, $\beta = (y_1, y_2, \ldots\ldots, y_n)$ in $\mathbf{C}^n$. Define $\langle\alpha, \beta\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$. **5+5+4**

Show that <, > is an inner product.

(b) Prove that an orthogonal set of non-zero vectors is linearly independent. Discuss the converse.

(c) Find the orthonormal basis for the subspace spanned by the vectors : (3, 0, 4) , (−1, 0, 7) and (2, 9, 11).

5. (a) Define eigen value and eigen vector of a linear transformation. Show that eigen vectors corresponding to distinct eigen values are linearly independent.

(b) Verify the Cayley-Hamilton theorem for the following matrix : **5+5+4**

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

(c) Check the diagonalizability of the matrix :

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

6. (a) Find the rank and nullity of the matrix : **5+5+4**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(b) Find the basis for the range space of the matrix :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) Define hyper space and give example.

7. (a) Let W be the subspace of an inner product V spanned by an orthonormal set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $\beta \in V$, then prove the following **7+7**

$$\alpha = \sum_k \frac{<\beta, \alpha_k>}{||\alpha_k||^2} \cdot \alpha_k$$

is the (unique) best approximation to β by vectors in W.

(b) Determine all possible Jordan Canonical forms for a linear operator $T: V \rightarrow V$, whose characteristic polynomial is $(x-2)^3(x-5)^2$.

8. (a) State and prove Bessel's inequality. **7+7**

(b) Solve the system of equations : $\frac{dx}{dt} = 3x - 10y$

$$\frac{dy}{dt} = x - 4y$$

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