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21MAT2C7L

Sl. No.

M.Sc. II Semester Degree Examination, October - 2023

MATHEMATICS

Measure Theory and Integration

NEP

Time : 3 Hours Maximum Marks : 70

Note : *Answer **any five** questions with question no.**1 Compulsory**.*

1. (a) If $\{A_n\}$ is a countable family of subsets of **R**, then show that $m^*\left[\bigcup_{i=1}^{\infty} A_n\right] \leq \sum_{n=1}^{\infty} m^*(A_n)$ **5**

(b) Prove or disprove : The outer measure of an interval is equal to its length. **5**

(c) Show that the Cantor's set is measurable and its measure is zero. **4**

2. (a) Prove the equivalence of following definitions of measurable function. **5**

(i) E [f>a] (ii) E [f⩾a] (iii) E [f<a] (iv) E [f⩽a]

(b) Show that : A continuous function defined over a measurable set E is measurable. **4**

(c) Define equivalent functions. Also prove that if one of the equivalent functions is measurable then another is measurable. **5**

3 (a) Prove that a Riemann integrable function defined on [a, b] is measurable. Discuss the converse. **6**

(b) If f is a bounded function in L[a, b] and c ∊ **R** then show that c*f* ∊ L[a, b] and $\int_a^b c f = c\int_a^b f$ **4**

(c) Let ϕ and ψ be simple functions which vanish outside a set E of finite measure then show that for a, b ∊ **R**, $\int(a\phi+b\psi) = a\int\phi + b\int\psi$ **4**

4. (a) State and prove that Lebesgue monotonic convergence theorem. **5**

(b) Prove or disprove : Every absolutely continuous function *f* defined on [a, b] is of bounded variation. **5**

(c) Verify the result of bounded convergence theorem for the function **4**

$$f_n(x) = \frac{nx}{1+n^2x^2}, 0 \leq x \leq 1.$$

5. (a) State and prove that Cauchy-Schwarz' inequality. **6**

(b) The space (L^p, d), $p > 1$ is a metric space. **4**

(c) Using Fubini's theorem, verify, **4**

$$\int_0^1 \left\{ \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dx \right\} dy \neq \int_0^1 \left\{ \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dy \right\} dx$$

6. (a) Let f be a measurable function defined over a measurable set E and $c \in \mathbf{R}$ then prove that functions cf, $f+c$, $|f|$, f^2 are measurable. **5**

(b) Let f be bounded real valued measurable function defined over a measurable set E of finite measure such that $a \leq f(x) \leq b$ then prove $a.m(E) \leq \int_E f \leq b.m(E)$ **4**

(c) If $\int_A f(x)dx = 0$ for every measurable subset A of a measurable set E then show that $f(x) = 0$ a.e on E. **5**

7. (a) Prove or disprove : Every absolutely continuous function is an indefinite integral of its own derivate. **5**

(b) Let $f(x) = \begin{cases} 0; \text{ for } x=0 \\ x^{\frac{-1}{3}} : \text{ for } 0 < x \leq 1 \end{cases}$ **5**

Then evaluate L–integral of f defined on $E = [0, 1]$.

(c) If $f \in L^p$ [a, b] and $g \leq f$ then show that $g \in L^p$ [a, b], where $p>1$. **4**

8. (a) Show that : The set of all irrational numbers in [0, 1] is measurable and has measure 1. **5**

(b) If f and g are measurable function defined over a measurable set E then prove that $f \cup g$ and $f \cap g$ are measurable over E. **4**

(c) Let $<E_n>$ be a monotonically increasing sequence of measurable sets of $E = \bigcup_{r=1}^{\infty} E_r$ then prove that $m(E) = \lim_{n\to\infty} m(E_n)$. **5**

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