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M.Sc. II Semester Degree Examination, September/October - 2022

MATHEMATICS

21MAT2C7L-DSC 7 : Measure Theory and Integration

Time : 3 Hours Maximum Marks : 70

Note : *Answer **any five** of the following questions with question No. **1 (Q1) Compulsory**, Each question carries **equal** marks.*

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1. (a) If A is an algebra on X and $\{A_n\}$ is a sequence of elements in A, then prove that there exists a disjoint sequence $\{B_n\}$ in A such that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{k=1}^{\infty} B_k$ with $A_i \subset B_i$. **5**

(b) Prove - or - disprove : The outer measure of an interval is its length. **5**

(c) If E_1 and E_2 are measurable sets, then prove the following : $m(E_1 \cap E_2) = m(E_1) + m(E_2) - m(E_1 \cup E_2)$. **4**

2. (a) Let f and g be measurable functions on E, and c be any constant. Then show that the following functions are measurable. **5**

(i) $f + c$ (ii) $f + g$ (iii) f^4 (iv) $f.g$

(b) Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E. Prove that the following functions are measurable. **5**

(i) $\max\{f_1, f_2, ...f_n\}$ (ii) $\inf_n f_n$ (iii) $\overline{\lim} f_n$

(c) Define equivalent functions. Also prove that if one of the equivalent functions is measurable then another is also measurable. **4**

3. (a) Let $f : E \to \mathbf{R}$ be a simple function. Then prove the following : **5**

$$\alpha \int_{-E} f(x)\,dx = \int_E f = \alpha \overline{\int_E} f(x)\,dx.$$

(b) Prove that a Riemann integrable function defined on [a, b] is measurable. Discuss the converse. **5**

(c) Let f and g be two bounded and equivalent functions on E with $m(E) < \infty$. Then prove the following : **4**

$$\int_E f = \int_E g$$

Discuss the converse.

P.T.O.

4. (a) State and prove Fatou' lemma. Show by means of an example that this result cannot hold in case of general measurable functions. **5**

(b) Let $\{ U_n \}$ be a sequence of non - negative measurable functions, and let $f = \sum_{n=1}^{\infty} U_n$. Then prove that $\int f = \sum_{n=1}^{\infty} \int U_n$. **5**

(c) If f is a differentiable function on an Interval I, then prove that f' is measurable on I. **4**

5. (a) Let E = [0, 16] and f: E $\to$ **R** be a function defined by $f(x) = x^{-\frac{1}{4}}$. Then show that $f \in L^1(E)$, but $f \notin L^4(E)$. **5**

(b) State and prove Riesz - Minkowski's inequality. **9**

6. (a) State and prove Egoroff's theorem. **7**

(b) Let f and g be non - negative measurable functions defined on a set E with $f \leq g$ *a.e.* Then prove that $\int_E f \leq \int_E g$. **7**

7. (a) Evaluate the integral (Lebergue) of the function f : [0, 1] $\to$ **R** defined by **7**

$$f(x) = \begin{cases} \dfrac{1}{3\sqrt{x}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}.$$

(b) If f is an integrable function such that f = o a.e, then show that $\int f = 0$. **7**

8. (a) Show that outer measure of rational numbers is zero. **5**

(b) Prove that a continuous function on a measurable set is measurable. Discuss the converse. **5**

(c) Let f be bounded measurable function defined on E and let E_1 and E_2 be disjoint measurable subsets of E, then prove that $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$. **4**

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