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21MAT4C13L

Sl. No.

M.Sc. IV Semester Degree Examination, October - 2023

MATHEMATICS

Functional Analysis

(NEP)

Time : 3 Hours Maximum Marks : 70

Note : *Answer **any five** questions with question no. **1 compulsory**.*

1. (a) Let M be a closed linear subspace of normed linear space 'N'. If the norm of a coset $(x+M)$ in the quotient space 'N' defined by $\|x+M\| = \inf \{\|x+M\| : m \in M\}$ then show that N/M is a normed linear space. Further prove that 'N' is a Banach space then N/M is also Banach space.

(b) Let N and N' be normed linear space and T is a linear transformation from $N \to N'$, then prove that the following conditions are equivalent to one another.

(i) T is continuous

(ii) T is continuous at origin, i.e. $x_n \to 0 \Rightarrow T(x_n) \to 0$

(iii) $\exists$ a real number $k \geqslant 0$, such that $\|T(x)\| \leqslant k\|x\|$

(iv) If $S = \{x : \|x\| \leq 1\}$ is closed unit sphere in 'N' then its image T(S) is bounded in 'N'. **6+8**

2. (a) State and prove Reisz Lemma.

(b) State and prove Hahn-Banach theorem. **6+8**

3. (a) Let M be a closed linear subspace of a normed linear space and ϕ be a natural homomorphism onto the quotient N/M defined by $\phi(x) = x + M$. Prove that ϕ is a continuous for which $\|\phi\| \leq 1$.

(b) Let x, y be two normed Linear space and $T : x \to y$ be onto mapping then show that T^{-1} exists and is continuous if and only if there exist a constant $m > 0$ such that $m\|x\| \leq \|T(x)\| \forall x \in X$. **7+7**

P.T.O.

4. (a) State and prove Gram-Schmidth orthogonalization process.

(b) If S, S_1, S_2 are non empty subsets of a Hilbert space H, then prove that the following conditions holds.

(i) $\{0\}^{\perp} = H$ and $H^{\perp} = \{0\}$

(ii) $S \cap S^{\perp} \subseteq \{0\}$

(iii) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

(iv) $S^{\perp}$ is closed linear space of H

(v) $S \subseteq (S^{\perp})^{\perp} = S^{\perp\perp}$ **4+10**

5. (a) Prove that every non-zero Hilbert space contains a complete orthonormal set.

(b) If H is a Hilbert space then prove that there exist an antilinear norm preserving isometric isomorphism between H and H^*. **7+7**

6. (a) State and prove Closed Graph theorem.

(b) Show that an operator T on H is unitary if and only if T $(\{e_i\})$ is a complete orthonormal set. **7+7**

7. (a) If A_1 and A_2 are two self adjoint operators on H then show that $A_1 \cdot A_2$ is self adjoint if and only if $A_1 A_2 = A_2 A_1$.

(b) Show that a closed convex subset C of a Hilbert space 'H' contains a unique vector of smallest norm. **7+7**

8. (a) Show that an operator T on H is unitary if and only if it is an isometric isomorphism of H onto itself.

(b) Show that if $Tx = \lambda x$ then $T^* x = \bar{\lambda} x$.

(c) If P is a projection on H with range M and null space N, then show that $M \perp N$, if and only if P is a self adjoint and in the case $N = M^{\perp}$. **5+5+4**

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